We next use the fact that the Einstein tensor has a zero divergence as was shown in Chap. 5. Thus we have

$$G^{\alpha}{}_{\lambda\|\alpha} = G^{0}{}_{\lambda\|0} + G^{i}{}_{\lambda\|i} = 0 \tag{8.18}$$

If we now express the $G^{i}{}_{\lambda}$ in terms of the $G^{0}{}_{\lambda}$, we obtain in (8.18) a differential system for the $G^{0}{}_{\lambda}$ only. Indeed, carrying out the covariant differentiations in (8.18) and rearranging terms, we arrive at the form

$$G^{0}{}_{\lambda|0} = A^{\sigma i}{}_{\lambda} G^{0}{}_{\sigma|i} + B^{\sigma}{}_{\lambda} G^{0}{}_{\sigma} \tag{8.19}$$

where the coefficients $A^{\sigma i}{}_{\lambda}$ and $B^{\sigma}{}_{\lambda}$ depend only on the metric tensor and its first derivatives.

We have in (8.19) a system of four linear homogeneous partial differential equations of the first order for the four components $G^{0}{}_{\lambda}$. This system is already in normal form with respect to the variable x^0; indeed, the time derivatives of the unknown functions stand on the left, the functions and their spatial derivatives on the right. The coefficients of this system are by hypothesis continuous in the region considered. Hence the initial-value problem for this system possesses a unique solution for given data on S. In particular, we see that the initial data $G^{0}{}_{\lambda} = 0$ on S imply $G^{0}{}_{\lambda} \equiv 0$ since this is clearly a solution of our initial-value problem. Thus the general theory of the Cauchy problem for linear partial differential equations has led to the asserted dependence of the system (8.15*b*) upon the system (8.15*a*).

We have now obtained a clearer insight into the above-mentioned assertion of Hilbert about the four relations which must prevail between the 10 equations of the Einstein system (8.1). These relations are most clearly seen in the dependence of (8.15*b*) on (8.15*a*) and are an obvious consequence of the Bianchi identities and the related properties of the Einstein tensor.

Now the computational aspect of the system (8.15*a*) and (8.15*b*) should be quite obvious. We may prescribe on S initial data $g_{\alpha\beta}$ and $g_{\alpha\beta|\lambda}$ which are compatible with the condition $G^{0}{}_{\lambda} = 0$ on S. Next, we prescribe the four components $g_{\alpha 0}$ quite arbitrarily in time and space, subject only to the condition that they match the data on S. (See Sec. 2.4 on Gaussian coordinates.) For the sake of normalization, we demand also $g_{\alpha 0|0|0} = 0$ on S. With the $g_{\alpha 0}(x)$ so chosen, we return to the system

$$R_{ij} = \tfrac{1}{2} g^{00} g_{ij|0|0} + M_{ij} = 0 \tag{8.20}$$

where the M_{ij} depend on the given $g_{\alpha 0}(x)$ and their derivatives and on the unknowns g_{ij} and their derivatives. However, M_{ij} does not contain time derivatives of order greater than 1. We now have a proper initial-value problem for the $g_{ij}(x)$, which can be handled by the standard methods of the theory. The g_{ij} and $g_{\alpha 0}$ then lead to an Einstein tensor $G_{\alpha\beta}$, which satisfies $G^{0}{}_{\lambda} = 0$, as we have shown above.

Since $R_{ij} = 0$ is likewise fulfilled, we have shown that we can solve the initial-value problem locally while prescribing the unessential components $g_{\alpha 0}(x)$ quite arbitrarily.

The important insight gained by our analysis is that we cannot prescribe freely the metric on a spacelike hypersurface and obtain an evolution in time of these initial data by means of the Einstein equations. On the contrary, we have the inner compatibility conditions $G^{0}{}_{\lambda} = 0$ in the initial surface $x^0 = 0$. The Einstein equations impose these conditions on a three-space which is supposed to be empty of matter. On the other hand, once these conditions are fulfilled, an evolution can take place which can be calculated from the six Einstein equations $R_{ik} = 0$. These equations determine the development of the geometry in time within the arbitrary assignment of the $g_{0\lambda}$. They also guarantee that the compatibility conditions $G^{0}{}_{\lambda} = 0$ remain true once they are fulfilled for the time $x^0 = 0$. We have thus split the Einstein equation system into two parts with essentially different significance: (1) The condition that space be empty in the spacelike hypersurface $x^0 = \text{const}$ demands $G^{0}{}_{\lambda} = 0$. (2) The equations of time evolution of the geometry $R_{ik} = 0$ determine the future development of each compatible metric in the initial space.

The question of actually determining a solution g_{ij} with given initial data and chosen $g_{\alpha 0}$ belongs to the theory of the Cauchy problem and has great difficulties of its own. We have carried out the reduction of the rather involved field equations for the metric field in empty space to a situation where the purely mathematical investigation can proceed in standard fashion.

One can consider the Cauchy problem for the Einstein system to consist of two parts: We must first verify that the compatibility conditions $G^{0}{}_{\lambda} = 0$ are satisfied by the initial data on S. Then, in principle, we have only to solve the six second-order equations $R_{ij} = 0$ throughout the four-dimensional space since the first-order system $G^{0}{}_{\lambda} = 0$ will then be identically satisfied. But when one attempts to define a procedure to construct a solution as we did above, one has to rely on both the second-order system $R_{ij} = 0$ and the first-order system $G^{0}{}_{\lambda} = 0$ simultaneously. The proof of the existence and of the uniqueness of the solutions of Eqs. (8.15*a*) has been given under a certain simple differentiability hypothesis by Fourès-Bruhat (1952). A much easier proof can be given if one assumes analyticity of the solutions and uses the Cauchy-Kowalewsky theorem. But the assumption of analyticity is unnecessary and unnatural; indeed, we shall see in the next section that Einstein's equations are of hyperbolic type and thus do not require analytic solutions. A very clear discussion of this last point in the general case of second-order partial differential equations is given by Hadamard (1932).

8.4 Characteristic Hypersurfaces of the Einstein Equation System

As usual, the best insight into the nature of a system of differential equations will be obtained by studying those singular hypersurfaces for which the Cauchy initial-value problem cannot be solved without restriction. These are the characteristic hypersurfaces of the system. As was already pointed out in Chap. 4, along such hypersurfaces different solutions of the same equation system can meet continuously, and for this reason the characteristic hypersurfaces play the role of wave fronts in the propagation of physical phenomena and are the locus in space-time of signals carried by this phenomenon.

To find these singular hypersurfaces S, we must ask that at each point of S the determination of $g_{ij|0|0}$ in terms of the Cauchy data on S be impossible. This is clearly the case if and only if at each point of S we have $g^{00} = 0$. However, this description is not covariant and is valid only in a coordinate system such that the equation for S is $x^0 = \text{const}$. If we use another coordinate system $\bar{x}^\alpha$, we have, by the transformation law of tensors,

$$\bar{g}^{\alpha\beta}\frac{\partial x^0}{\partial \bar{x}^\alpha}\frac{\partial x^0}{\partial \bar{x}^\beta} = g^{00} = 0 \tag{8.21}$$

This is a partial differential equation for the characteristic hypersurface S which has the equation

$$x^0 = \varphi(\bar{x}^\alpha) = \text{const} \tag{8.22}$$

What is the geometric meaning of this characteristic equation? We observe that

$$\bar{\xi}_\alpha = \varphi_{|\alpha} = \frac{\partial x^0}{\partial \bar{x}^\alpha} \tag{8.23}$$

is a covariant vector. By virtue of (8.21), we have

$$\bar{g}^{\alpha\beta}\bar{\xi}_\alpha\bar{\xi}_\beta = 0 \tag{8.24}$$

that is, $\bar{\xi}_\alpha$ is a null vector. If $d\bar{x}^\alpha$ is a tangent vector to the characteristic surface S with the equation $\varphi = 0$, we have on S

$$d\varphi = \varphi_{|\alpha}\, d\bar{x}^\alpha = \bar{\xi}_\alpha\, d\bar{x}^\alpha = 0 \tag{8.25}$$

that is, $\bar{\xi}_\alpha$ is the normal vector to S. Thus the characteristic hypersurfaces S of the Einstein field equations are characterized by the fact that their normal vector is at every point a null vector. (We have already studied such null hypersurfaces in connection with their "one-way membrane" properties in Chap. 7.)

Consider next the vector $\bar{\xi}^\alpha$, the contravariant form of the normal vector $\bar{\xi}_\alpha$. A differential $d\bar{x}^\alpha$ in the direction $\bar{\xi}^\alpha$ is orthogonal to the normal vector by virtue of (8.24) and lies, therefore, in S. Thus we can also infer that S possesses a tangent null vector at every point. Through every point $\bar{x}$ of a hypersurface S we can draw the cone of vectors $d\bar{x}^\alpha$ which satisfy the equation $g_{\alpha\beta}\, d\bar{x}^\alpha\, d\bar{x}^\beta = 0$, the local light cone at $\bar{x}$. We see from the above that a characteristic hypersurface touches a local light cone at all its points since the differential $d\bar{x}^\alpha$ in the direction $\bar{\xi}^\alpha$ lies in S and also lies on the local light cone by virtue of (8.24).

8.5 Bicharacteristics of the Einstein System

We found in the preceding section that the characteristic surfaces S of the Einstein equations can be described in the form $\varphi(x^\alpha) = \text{const}$, where the function $\varphi(x^\alpha)$ satisfies the first-order partial differential equation

$$H(x^\mu,\varphi_{|\nu}) = g^{\alpha\beta}\varphi_{|\alpha}\varphi_{|\beta} = 0 \tag{8.26}$$

The general theory of surfaces which satisfy first-order partial differential equations asserts that such surfaces can be built up from a family of elements, the so-called strips, which depend only on one parameter. In order to give a precise definition of a strip, we first need the concept of a surface element. A surface element of S is characterized by its location in space-time x^α and by its normal vector $p_\alpha = \varphi_{|\alpha}$. A strip on the surface is then the one-parameter set $x^\alpha(\lambda)$, $p_\alpha(\lambda)$ such that $\varphi(x^\alpha(\lambda)) \equiv 0$ and $p_\alpha(\lambda) = \varphi_{|\alpha}(x^\beta(\lambda))$. Geometrically, a strip is a one-parameter set of surface elements which are laid out along a curve $x^\alpha(\lambda)$ on S.

There are many possible strips on a given surface. Each curve $x^\alpha(\lambda)$ on S gives rise to such a strip. Among all those strips there is the distinguished set of characteristic strips which can be determined by means of a system of ordinary differential equations without knowledge of the surface S. Thus, instead of solving the partial differential equation (8.26), we may build up the solution surface S by its characteristic strips, which can be obtained as the solution of ordinary differential equations. The theory of ordinary differential equations is considered more elementary than that of partial differential equations. Thus the reduction of the partial differential equation (8.26) to the theory of characteristic strips represents a mathematical simplification.

However, the idea of characteristic strips is also of great geometrical and physical significance. Indeed, we shall show that it follows from the general theory that two integral surfaces φ = const of (8.26) which have a surface element in common, i.e., are tangent to each other at one point, have the entire characteristic strip through that point in common. Hence different integral surfaces of our partial differential equations have characteristic strips in common; this property can be taken as the definition of the characteristic strips. Thus they are the elementary building blocks of the integral surfaces. If we interpret the characteristic hypersurfaces of a partial differential system as the wave fronts of perturbations or signals, their characteristics in turn describe the propagation of localized perturbations along curves, i.e., rays of the propagating phenomenon. The characteristic curves on the characteristic hypersurfaces of the original system of partial differential equations are called the *bicharacteristics* of the original system.

To determine the characteristic strips on the integral surface S we proceed as follows: We assume, for the time being, that the solution $\varphi(x^\alpha)$ of (8.26) is known and define the curves $x^\alpha(\lambda)$ by means of the system of ordinary first-order differential equations

$$\dot{x}^\alpha(\lambda) = \frac{\partial H}{\partial \varphi_{|\alpha}} = 2g^{\alpha\beta}\varphi_{|\beta} \tag{8.27}$$

whose right side depends only on the $x^\alpha(\lambda)$. The integral curves of (8.27) will lie on the surface S, that is, satisfy $\varphi(x^\alpha)$ = const. Indeed, we have, by virtue of (8.26) and (8.27),

$$\frac{d}{d\lambda}\varphi(x^\alpha(\lambda)) = \varphi_{|\alpha}\dot{x}^\alpha = 2g^{\alpha\beta}\varphi_{|\alpha}\varphi_{|\beta} = 0 \tag{8.28}$$

The curves $x^\alpha(\lambda)$ defined by (8.27) now give rise to the strips $x^\alpha(\lambda)$, $p_\alpha(\lambda)$ on S, where

$$p_\alpha(\lambda) = \varphi_{|\alpha}(x^\mu(\lambda)) \tag{8.29}$$

We compute from (8.29) and (8.27)

$$\dot{p}_\alpha(\lambda) = \varphi_{|\alpha|\mu}\dot{x}^\mu = \frac{\partial H}{\partial \varphi_{|\mu}}\varphi_{|\alpha|\mu} \tag{8.30}$$

On the other hand, we may differentiate the identity (8.26) with respect to x^α and find

$$\frac{\partial H}{\partial x^\alpha} + \frac{\partial H}{\partial \varphi_{|\mu}}\varphi_{|\alpha|\mu} = 0 \tag{8.31}$$

Thus (8.30) simplifies to

$$\dot{p}_\alpha(\lambda) = -\frac{\partial H}{\partial x^\alpha} \tag{8.32}$$

At this point we can drop the assumption that the solution $\varphi(x^\alpha)$ of (8.26) is known. For (8.27) and (8.32) form a consistent ordinary differential system

$$\dot{x}^\alpha(\lambda) = \frac{\partial H(x^\alpha,p_\alpha)}{\partial p_\alpha} \qquad \dot{p}_\alpha(\lambda) = -\frac{\partial H(x^\alpha,p_\alpha)}{\partial x^\alpha} \tag{8.33}$$

which depends only on the known function

$$H(x^\alpha,p_\alpha) = g^{\alpha\beta}p_\alpha p_\beta \tag{8.34}$$

and can be integrated without knowledge of the solution $\varphi(x^\alpha)$ of the partial differential equation (8.26). The set of values $x^\alpha(\lambda)$, $p_\alpha(\lambda)$ is a characteristic strip on S; (8.33) is the equation system for such a characteristic strip. S can be built up from a manifold of such characteristic strips, depending on a sufficient number of parameters.

A characteristic strip is determined by the initial values $x^\alpha(0)$, $p_\alpha(0)$ and the system (8.33). Hence, if a strip has one element $x^\alpha(0)$, $p_\alpha(0)$ in common with a surface S, it will lie in it for all λ values, and if two surfaces S have a surface element in common, they must share the entire characteristic strip through it in common also, as we asserted at the beginning of this section.

In order to determine the bicharacteristics from the differential system (8.33), we draw on the well-known formalism of Hamiltonian mechanics. We may interpret the $x^\alpha(\lambda)$ and $p_\alpha(\lambda)$ as conjugate canonical variables in a mechanical system. We introduce the Lagrange function

$$L(x^\alpha,\dot{x}^\alpha) = \dot{x}^\alpha p_\alpha - H(x^\alpha,p_\alpha) \tag{8.35}$$

in which the p_α have been eliminated through the system of implicit equations

$$\dot{x}^\alpha = \frac{\partial H(x^\mu, p_\mu)}{\partial p_\alpha} \tag{8.36}$$

We then have the identities

$$\frac{\partial L}{\partial \dot{x}^\alpha} = p_\alpha \qquad \frac{\partial L}{\partial x^\alpha} = -\frac{\partial H}{\partial x^\alpha} \tag{8.37}$$

which are an immediate consequence of (8.35) and (8.36). By virtue of (8.37) the system (8.33) goes over into the Lagrangian form

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha} \tag{8.38}$$

Thus, we may globally characterize the bicharacteristics of our problem through a variational problem for which Eqs. (8.38) are the Euler-Lagrange equations. We find: The bicharacteristics of the Einstein equations are the extremals of the variational problem

$$\delta \int L \, d\lambda = 0 \tag{8.39}$$

with the Lagrange function L corresponding to the Hamiltonian (8.34). We find

$$L = \dot{x}^\alpha p_\alpha - g^{\alpha\beta} p_\alpha p_\beta \tag{8.40}$$

where we have to eliminate the p_α through the linear system

$$\dot{x}^\alpha = 2g^{\alpha\beta} p_\beta \tag{8.41}$$

This leads to

$$p_\alpha = \tfrac{1}{2} g_{\alpha\beta} \dot{x}^\beta \tag{8.42}$$

Furthermore, from the definition (8.34) of H in terms of p_α and the above, we obtain

$$H(x^\alpha, p_\alpha) = g^{\alpha\beta} p_\alpha p_\beta = \tfrac{1}{2} \dot{x}^\alpha p_\alpha = \tfrac{1}{4} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{8.43}$$

Therefore L is given as a function of x^α and $\dot{x}^\alpha$ by

$$L = \tfrac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta - \tfrac{1}{4} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \tfrac{1}{4} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = H \tag{8.44}$$

Thus the bicharacteristics are simply the geodesics of our metric

$$\delta \int g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \, d\lambda = 0 \tag{8.45}$$

which must, moreover, satisfy the side condition

$$g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0 \tag{8.46}$$

as a consequence of the differential equation (8.26), $H = 0$, and (8.44). Thus *the bicharacteristics of the Einstein equations are the null geodesics.*

The bicharacteristics of the Einstein equations for the metric tensor $g_{\alpha\beta}$ represent the lines along which perturbations of the metric can propagate. Since the metric determines the gravitational fields, we may say that gravitational perturbations spread along null geodesics. These are the same curves along which electromagnetic effects are propagated. Thus gravitational and electromagnetic phenomena have similar propagation properties, and the importance of the null geodesics for all wave phenomena is evident.

Our result on gravitational rays illustrates how a purely mathematical analysis of the structure of a differential system may lead to important physical insights and very significant results.

8.6 Uniqueness Problem for the Einstein Equations

As was mentioned in Sec. 8.3, Mme. Fourès-Bruhat has given an existence and uniqueness proof for the solution of the field equations of general relativity theory under very general assumptions. We shall restrict ourselves to a much simpler situation. We shall deal with the case of a static universe which is entirely free of matter and shall prove that it must possess a constant Lorentz metric throughout, if we demand that it tend to a Lorentzian universe at infinity. Since the constant Lorentz metric is obviously a solution of the Einstein field equations $R_{\mu\nu} = 0$ for the empty universe, we are dealing with a uniqueness problem for these field equations. This particular uniqueness proof is due to Lichnerowicz (1955). Its significance for the physical aspects of general relativity lies in the fact that it displays clearly the importance of the boundary conditions at infinity, i.e., the behavior of the universe at large. It shows that the local solution is strongly influenced by the global behavior of the metric and explains the central role of cosmological theories in general relativity.

We start out by precisely defining what we understand by a static space-time manifold (see Sec. 6.1 and Prob. 6.6). Such a manifold is characterized by the fundamental metric form

$$ds^2 = \xi^2(dx^0)^2 + g_{ij}\,dx^i\,dx^j \tag{8.47}$$

where the coefficients ξ and g_{ij} depend only on the three-space variables x^i and not on the time variable x^0. Since we demand the usual signature of a relativistic space-time metric, we must assume that the form $g_{ij}\,dx^i\,dx^j$ is negative-definite. Such a static line element was already considered in the case of the Schwarzschild metric; however, at present we do not make any assumption on spherical symmetry in space.

One may visualize a static space-time manifold as built up of identical layers of three-dimensional space hypersurfaces which are all orthogonal to the time-coordinate lines. Since the metric is independent of the time coordinate, all these three-dimensional spaces are isometric with the three-dimensional metric tensor g_{ij}, and if we disregard the unessential time coordinate x^0, we may identify them all with the "base space" $(x^0 = 0, x^i)$. In this base space, we now have the tensor g_{ij} and the scalar field $\xi(x^i)$, which are combined through the field equations $R_{\mu\nu} = 0$.

Our problem is to find a solution system $\{g_{ij}, \xi(x^i)\}$ of these differential equations which is twice continuously differentiable in the entire base space and tends to the system $\{-\delta_{ij},1\}$ for x^i becoming infinite. We wish to show that these boundary conditions have the only solution

$$g_{ij}(x^k) \equiv -\delta_{ij} \qquad \xi(x^i) \equiv 1 \tag{8.48}$$

In order to study this uniqueness problem, we observe that the coordinate x^0 plays a distinguished role in our formulas. We shall use this fact to simplify the Einstein field equations and to express them in terms of the tensors and Christoffel symbols of the base space of the x^i. In order to distinguish the Riemann tensor and Christoffel symbols of the three-dimensional base space from the analogous quantities of the space-time manifold, we shall denote the three-dimensional base-space quantities by an asterisk.

We start out with the form of the metric tensor and its inverse

$$g_{\alpha\beta} = \begin{pmatrix} \xi^2 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij} & \\ 0 & & & \end{pmatrix} \qquad g^{\alpha\beta} = \begin{pmatrix} \xi^{-2} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g^{ij} & \\ 0 & & & \end{pmatrix} \tag{8.49}$$

Next we compute the four-dimensional Christoffel symbols

$$\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} = g^{\alpha\tau}[\beta\gamma,\tau] = \tfrac{1}{2}g^{\alpha\tau}(g_{\beta\tau|\gamma} + g_{\gamma\tau|\beta} - g_{\beta\gamma|\tau}) \tag{8.50}$$

For all spatial indices we have, clearly,

$$\begin{Bmatrix} i \\ k\ l \end{Bmatrix} = \begin{Bmatrix} i \\ k\ l \end{Bmatrix}^* \tag{8.51}$$

since all g_{ik} are time-independent and all g_{i0} vanish. We next have to consider the Christoffel symbols with at least one zero index. Because of (8.49), we have

$$\begin{Bmatrix} 0 \\ \beta\ \gamma \end{Bmatrix} = \frac{1}{2\xi^2}(g_{\beta 0|\gamma} + g_{\gamma 0|\beta}) \tag{8.52}$$

which leads to the three cases

$$\begin{Bmatrix} 0 \\ i\ k \end{Bmatrix} = 0 \qquad \begin{Bmatrix} 0 \\ i\ 0 \end{Bmatrix} = \frac{\xi_{|i}}{\xi} \qquad \begin{Bmatrix} 0 \\ 0\ 0 \end{Bmatrix} = 0 \tag{8.53}$$

Similarly,

$$\begin{Bmatrix} \alpha \\ 0\ \gamma \end{Bmatrix} = \tfrac{1}{2}g^{\alpha\tau}(g_{0\tau|\gamma} - g_{0\gamma|\tau}) \tag{8.54}$$

which gives the additional cases

$$\begin{Bmatrix} i \\ 0\ k \end{Bmatrix} = 0 \qquad \begin{Bmatrix} i \\ 0\ 0 \end{Bmatrix} = -\tfrac{1}{2}g^{il}g_{00|l} = -g^{il}\xi_{|l}\xi \tag{8.55}$$

From these formulas we now compute the values of the contracted Riemann tensor:

$$R_{\alpha\beta} = \begin{Bmatrix} \rho \\ \beta\ \rho \end{Bmatrix}_{|\alpha} - \begin{Bmatrix} \rho \\ \alpha\ \beta \end{Bmatrix}_{|\rho} + \begin{Bmatrix} \rho \\ \alpha\ \sigma \end{Bmatrix}\begin{Bmatrix} \sigma \\ \beta\ \rho \end{Bmatrix} - \begin{Bmatrix} \rho \\ \alpha\ \beta \end{Bmatrix}\begin{Bmatrix} \sigma \\ \sigma\ \rho \end{Bmatrix} \tag{8.56}$$

By virtue of (3.11), we have

$$\begin{Bmatrix} \rho \\ \beta\ \rho \end{Bmatrix} = (\log\sqrt{-g})_{|\beta} = (\log\xi + \log(\sqrt{-g^*}))_{|\beta} \tag{8.57}$$

and hence, because of the independence of our metric on x^0,

$$\left\{ {\rho \atop 0\ \rho} \right\} = 0 \tag{8.58}$$

We conclude from (8.51), (8.53), and (8.55) that

$$\begin{aligned} R_{ik} &= (\log \xi + \log \sqrt{-g^*})_{|i|k} - \left\{ {r \atop i\ k} \right\}^*_{|r} + \left\{ {r \atop i\ s} \right\}^* \left\{ {s \atop k\ r} \right\}^* \\ &\quad + \left\{ {0 \atop i\ 0} \right\} \left\{ {0 \atop k\ 0} \right\} - \left\{ {r \atop i\ k} \right\}^* (\log \xi + \log \sqrt{-g^*})_{|r} \\ &= R^*_{ik} + \frac{1}{\xi}\left(\xi_{|i|k} - \left\{ {r \atop i\ k} \right\}^* \xi_{|r} \right) \end{aligned} \tag{8.59}$$

To simplify the above relation we introduce the covariant vector in three-space,

$$\xi_i = \xi_{|i} \tag{8.60}$$

and its covariant derivative in the metric of the base space,

$$\xi_{i\|k} = \xi_{|i|k} - \left\{ {r \atop i\ k} \right\}^* \xi_r \tag{8.61}$$

In this notation we may express

$$R_{ik} = R^*_{ik} + \frac{1}{\xi}\, \xi_{i\|k} \tag{8.62}$$

as the relation between the contracted Riemann tensors in the two metrics.

Similarly, by (8.58), (8.53), and (8.55),

$$R_{i0} = 0 \tag{8.63}$$

Finally, we can write R_{00} by means of the above identities in the form

$$\begin{aligned} R_{00} = &- \left\{ {r \atop 0\ 0} \right\}_{|r} + 2 \left\{ {0 \atop 0\ r} \right\} \left\{ {r \atop 0\ 0} \right\} \\ &- \left\{ {r \atop 0\ 0} \right\} ((\log \xi)_{|r} + (\log \sqrt{-g^*})_{|r}) \end{aligned} \tag{8.64}$$

We introduce the contravariant form of the vector ξ_i in the base space

$$\xi^i = g^{il}\xi_{|l} \tag{8.65}$$

Then, by (8.53) and (8.55), we can transform (8.64) into

$$\begin{aligned} R_{00} &= (\xi\xi^r)_{|r} - \xi_r\xi^r + \xi\xi^r(\log \sqrt{-g^*})_{|r} \\ &= \xi(\xi^r{}_{|r} + \xi^r(\log \sqrt{-g^*})_{|r}) = \xi\, \nabla^{*2}\xi \end{aligned} \tag{8.66}$$

Here we have used the identity (3.12),

$$\xi^r{}_{\|r} = \operatorname{div}^* \xi^r = \frac{1}{\sqrt{-g^*}} (\xi^r \sqrt{-g^*})_{|r} = \xi^r{}_{|r} + \xi^r(\log \sqrt{-g^*})_{|r} \tag{8.67}$$

and the definition of the Laplace operator ∇^{*2} in the base space in terms of the metric g_{ij}:

$$\nabla^{*2}\xi = \operatorname{div}^* \operatorname{grad} \xi = (g^{rs}\xi_{|s})_{\|r} \tag{8.68}$$

Thus the Einstein equations reduce to the following differential system in three-dimensional space:

$$\nabla^{*2}\xi = 0 \tag{8.69}$$

and

$$R^*_{ik} + \frac{1}{\xi}\, \xi_{i\|k} = 0 \tag{8.70}$$

We assume, of course, that $\xi(x^i) \neq 0$ since we wish to deal with a regular metric in space-time.

The procedure of solving the equation system (8.69) and (8.70) is now obvious. We shall first solve the Laplace equation (8.69) for $\xi(x^i)$, with the requirement that ξ be twice continuously differentiable in space and tend uniformly to 1 at infinity. Having determined $\xi(x^i)$ from these conditions, we insert it into (8.70) and determine g_{ij} from these equations.

If ∇^* were the ordinary Laplace operator, it would be evident that $\xi(x^i) \equiv 1$. Indeed, the solutions of the classical Laplace equation, the so-called harmonic functions, satisfy the maximum-minimum principle; i.e., in the neighborhood of each point P of regularity, there are points where the function takes values larger and smaller than at that point P itself. It follows that in each domain of regularity, the maximum and the minimum of the function are attained on the boundary of the domain. Since we assume that our solution is regular at every finite point and tends to the value 1 at infinity, it is clear that the maximum and the minimum of the solution must both be 1 and hence $\xi(x^i) \equiv 1$. We shall give in the next section a proof due to Hopf (1927), which extends the maximum-minimum principle to the generalized Laplacian needed here. Hence we may conclude from this principle that the differential equation (8.69) and the boundary conditions at infinity imply

$$\xi(x^i) \equiv 1 \tag{8.71}$$

If we insert the value (8.71) for ξ into (8.70), we find

$$R^*_{ik} = 0 \tag{8.72}$$

It is now a remarkable fact, which is of interest in its own right, that a three-dimensional space whose contracted Riemann tensor R^*_{ik} vanishes is a flat space; that is, its full Riemann tensor R^*_{iklm} is identically zero. Indeed, because of the numerous antisymmetries of the Riemann tensor, there are only a few nonzero independent components of the full Riemann tensor in three-space:

$$\begin{matrix} R^*_{12\,12} & R^*_{12\,13} & R^*_{12\,23} \\ R^*_{13\,13} & R^*_{13\,23} & R^*_{23\,23} \end{matrix} \tag{8.73}$$

Thus the number of independent components of the full Riemann tensor is precisely equal to six, which is also the number of the independent components of the symmetric contracted Riemann tensor. On the other hand, we can express by

$$R^*_{ik} = g^{lm} R^*_{milk} \tag{8.74}$$

the R^*_{ik} components as linear combinations of the six components of R^*_{milk}.

It is easily verified that the determinant of the linear transformation (8.74) is in general nonzero and that, conversely, the R^*_{milk} can be expressed linearly in terms of the R^*_{ik}. Thus Eqs. (8.72) imply indeed that R^*_{milk} vanishes identically. As we showed in Sec. 5.2, this implies that the base space is pseudo-Euclidean, and since we demand at infinity the limit condition $g_{ij} \to -\delta_{ij}$, we see that (8.48) is indeed fulfilled throughout the entire base space.

Thus, under the assumption of the maximum-minimum principle for the generalized Laplace equation (8.69), we have proved the uniqueness of the Lorentz metric for an empty universe which becomes Lorentzian at spatial infinity.

8.7 The Maximum Principle for the Generalized Laplace Equation

We give, for the sake of completeness, the Hopf argument for the maximum-minimum principle for equations of the form

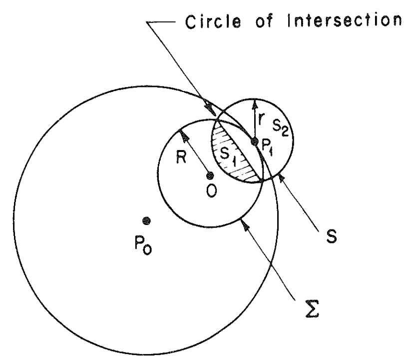
$$L[\xi] = g^{ik}\frac{\partial^2 \xi}{\partial x^i \, \partial x^k} + A^i \frac{\partial \xi}{\partial x^i} = 0 \tag{8.75}$$

with a negative-definite coefficient matrix g^{ik}. Clearly, our particular Laplace equation (8.69) is of this standard form. The proof is quite simple, and the result is of sufficient importance for many problems in applied mathematics and mathematical physics to justify its inclusion in this book.

Let us suppose that $\xi(x^i)$ is a solution of (8.75), defined in the entire space; assume it takes its minimum, say m, at some finite set of points. We select an arbitrary point P_0, where $\xi > m$, and consider that sphere around P_0 inside of which $\xi > m$, but on the surface of which the equation $\xi = m$ is fulfilled at one or more points. Such a sphere surely exists by our assumption. We next construct a sphere Σ which lies entirely inside this first sphere but which is tangent to it at a point P_1 at which $\xi = m$. Thus we can assert that inside and on the boundary of Σ, it is true that $\xi > m$, except for the one distinguished point P_1 at which $\xi = m$. The sphere Σ may furthermore be supposed centered at the origin and of radius R (Fig. 8.1).

We next consider one more sphere S with center P_1 and radius $r < R$. It intersects the sphere Σ, and its surface will be divided by Σ into two

Fig. 8.1



parts. We denote by S_1 that part of the surface S which is contained in Σ, including the curve of intersection (a circle), and by S_2 the open-surface region of S which is completely outside of Σ. Since $\xi(x^i)$ is continuous in the closed domain S_1, it has there a minimum and, by construction, this minimum is strictly greater than m. On S_2 we can only assert $\xi \geq m$, by our basic assumption. Thus there exists a positive number δ such that

$$\begin{aligned} &\xi(x^i) \geq m + \delta \qquad \text{on } S_1 \qquad \delta > 0 \\ &\xi(x^i) \geq m \qquad \text{on } S_2 \end{aligned} \tag{8.76}$$

Following Hopf, we introduce the function

$$h(x^i) = e^{-\alpha R^2} - e^{-\alpha r^2} \qquad \alpha > 0 \qquad r^2 = x^i x_i \tag{8.77}$$

where we define, for notational convenience, $x_i = \delta_{ij}x^j = x^j$. On Σ we have $h(x^i) = 0$, and everywhere

$$-1 < h(x^i) < 1 \tag{8.78}$$

A straightforward calculation yields

$$L[h] = e^{-\alpha r^2}(-4\alpha^2 g^{ik}x_i x_k + 2\alpha g^{ik}\delta_{ik} + 2\alpha A^i x_i) \tag{8.79}$$

Since g^{ik} is negative-definite, it is clear that we can choose α sufficiently large so that $L[h] > 0$ inside and on S. Since L is a linear operator on ξ, it follows also that, by virtue of (8.75),

$$L[\xi + \lambda h] > 0 \qquad \text{in and on } S \tag{8.80}$$

for all values $\lambda > 0$. Let us choose $0 < \lambda < \delta$, where δ is defined preceding (8.76).

We recall the bounds (8.78) for $h(x^i)$, and the fact that

$$\begin{aligned} &h(x^i) \leq 0 \qquad \text{on } S_1 \\ &h(x^i) > 0 \qquad \text{on } S_2 \end{aligned} \tag{8.81}$$

and also the inequalities (8.76). We conclude that, since $0 < \lambda < \delta$,

$$\begin{aligned} &\xi + \lambda h > m \qquad \text{on } S_1 \\ &\xi + \lambda h > m \qquad \text{on } S_2 \\ &\xi + \lambda h = m \qquad \text{at } P_1 \end{aligned} \tag{8.82}$$

Thus $\xi + \lambda h$ is a nonconstant, twice continuously differentiable function which must have a minimum inside of S, say at P_2. Hence, at P_2, the necessary minimum condition must hold:

$$\frac{\partial}{\partial x^i}(\xi + \lambda h)\Big|_{P_2} = 0 \tag{8.83}$$

At P_2 the differential operator $L[\xi + \lambda h]$ becomes very simple, since all first derivatives of the argument function vanish. By (8.80), we can assert that

$$L[\xi + \lambda h] = g^{ik}\frac{\partial^2}{\partial x^i\,\partial x^k}(\xi + \lambda h)\Big|_{P_2} > 0 \tag{8.84}$$

On the other hand, we know that g^{ik} is a negative-definite matrix, while the well-known necessary condition for the minimum of $\xi + \lambda h$ at P_2 implies

$$Q(t^i) = \frac{\partial^2}{\partial x^i\,\partial x^k}(\xi + \lambda h)\Big|_{P_2} t^i t^k \geq 0 \tag{8.85}$$

for all real values t^i. We can bring the symmetric quadratic form $Q(t^i)$ onto principal axes; i.e., there exist linear forms

$$\tau^j = \alpha^j{}_i t^i \tag{8.86}$$

such that $Q(t^i)$ takes the form

$$Q(t^i) = \sum_{j=1}^{3} \lambda_j(\tau^j)^2 = \sum_{j=1}^{3} \lambda_j \alpha^j{}_i \alpha^j{}_k t^i t^k \tag{8.87}$$

with nonnegative eigenvalues λ_j. Hence, comparing the coefficients of $t^i t^k$ in (8.85) and (8.87), we can assert

$$\frac{\partial^2}{\partial x^i\,\partial x^k}(\xi + \lambda h)\Big|_{P_2} = \sum_{j=1}^{3} \lambda_j \alpha^j{}_i \alpha^j{}_k \tag{8.88}$$

and (8.84) becomes

$$\sum_{j=1}^{3} \lambda_j (g^{ik}\alpha^j{}_i \alpha^j{}_k) > 0 \tag{8.89}$$

This contradicts the fact that g^{ik} is negative-definite and the λ_j are nonnegative. Thus our assumption regarding a finite minimum point leads to an obvious contradiction. If we replace ξ by $-\xi$, we see at once that a finite maximum point is likewise excluded. Thus the proof on the nonexistence of a finite maximum or minimum within the domain of definition is complete.

Exercises

8.1 In special relativity Maxwell's equations for free space may be written in terms of the four-vector potential A^μ as

$$\Box^2 A^\mu = g^{\nu\alpha} A^\mu{}_{|\nu|\alpha} = 0$$

if we impose the Lorentz condition $A^\mu{}_{|\mu} = 0$ (see Exercise 4.4). Formulate the initial-value problem for this system.

8.2 (*continued*) Prove that

$$\nabla^2 A^0 = \frac{\partial}{\partial t} \nabla \cdot \mathbf{A}$$

so that the component A^0 is not the solution of a typical Cauchy problem of second-order differential equations.

8.3 Must all partial differential equations or systems of partial differential equations possess characteristic surfaces? If an equation or system of equations does not possess characteristic surfaces, what sort of physical process could it describe, and what sort of physical process could it not describe?

8.4 The Schrödinger equation, which describes the motion of a free particle in nonrelativistic quantum theory, is of the form

$$i\lambda \frac{\partial \psi}{\partial t} + \nabla^2 \psi = 0 \qquad \psi = \psi(\mathbf{x},t)$$

where λ is a real constant. [$|\psi(\mathbf{x},t)|^2$ is interpreted as the relative probability of the particle's being at position $\mathbf{x}$ at time t.] Discuss the Cauchy problem and the existence of characteristic surfaces for this equation.

8.5 The Klein-Gordon equation replaces the Schrödinger equation in the relativistic quantum theory of mesons. It has the form, in flat space,

$$\Box^2 \psi + \tau^2 \psi = 0$$

where τ is a real constant. Discuss the Cauchy problem and the existence of characteristic surfaces for this equation.

8.6 How would the study of differential equations change if we considered equations of higher than second order? What Cauchy data would be needed? Would characteristic surfaces occur?

8.7 Obtain the characteristic surfaces of Maxwell's equations in flat space. You may wish to consult Chap. 4 for one method of doing this. Obtain also the bicharacteristics and show that they represent null geodesics, physically interpreted as light rays. Illustrate this in a picture with the z coordinate suppressed, analogous to Fig. 7.1.

8.8 Draw analogies between the mathematical structure of Maxwell's equations in flat space and the Einstein equations, and between the characteristics and bicharacteristics of the two systems of equations.

8.9 Show how the uniqueness theorem of Sec. 8.6 breaks down if the function ξ is allowed to have a singularity.

Problems

8.1 Investigate the characteristics and bicharacteristics of the Maxwell equations in a general Riemann space.

8.2 In the cosmological problem we do not deal with Euclidean boundary conditions at infinity (see Chap. 12). One of the possible forms taken by the metric is, in Cartesian coordinates,

$$ds^2 = c^2\, dt^2 - R(t)^2[dx^2 + dy^2 + dz^2]$$

Analyze the uniqueness problem for this metric form instead of that discussed in the text, (8.47).

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CHAPTER

9

The Linearized Field Equations

The field equations for free space which we developed in Chap. 5 are clearly not linear, so the superposition principle does *not* hold; that is, if $g_{\alpha\beta}$ and $g^*_{\alpha\beta}$ are solutions to the field equations, then a linear combination of the two is *not* necessarily a solution. The physical reason for this is easily understood, for the gravitational field of a body can do work and must therefore contain energy. Since it possesses energy, it must possess effective mass, and thereby create a *further* gravitational field. That is, the field *itself* can serve as part of *its own source*. Because of this feedback effect the gravitational field produced by two bodies is not a simple sum of the separate fields of the two bodies, but involves the detailed structure of the interacting fields.

In this chapter we shall develop a set of approximate linear equations in which the feedback effect of the gravitational field as its own source is ignored. In such an approximate theory gravitational effects are considered to be simply additive, as they are in the classical gravitational theory. Furthermore, because of their linearity, these equations will possess the virtue of being mathematically simpler than the exact gravitational field equations of Chap. 5.

9.1 Linearization oı the Field Equations

In order that we may ignore the feedback effect of the gravitational field as part of its own source, it is clear that we must assume that the field is weak. Thus we shall deal throughout this chapter with a metric tensor which differs only slightly from the flat-space metric tensor. Accordingly, we may write the metric tensor as the flat-space Lorentz metric tensor $\eta_{\alpha\beta}$ plus a perturbation term $\epsilon\gamma_{\alpha\beta}$,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon\gamma_{\alpha\beta} \tag{9.1}$$

and consider only first-order terms in the parameter ϵ as significant in all equations.

It will be convenient in this chapter to use the Minkowski coordinates ict, x, y, and z in place of the usual coordinates of special relativity. In this coordinate system the Lorentz metric tensor has the simple form

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{9.2}$$

that is, $\eta_{\alpha\beta}$ is simply the negative of the Kronecker delta $\delta_{\alpha\beta}$.

Using the Minkowski coordinates and the form given in (9.2) for $\eta_{\alpha\beta}$, let us consider the free-space gravitational field equations (5.119):

$$0 = R_{\eta\lambda} = \left\{ {\beta \atop \beta\ \eta} \right\}_{|\lambda} - \left\{ {\beta \atop \lambda\ \eta} \right\}_{|\beta} + \left\{ {\beta \atop \tau\ \lambda} \right\} \left\{ {\tau \atop \beta\ \eta} \right\} - \left\{ {\beta \atop \tau\ \beta} \right\} \left\{ {\tau \atop \lambda\ \eta} \right\} \tag{9.3}$$

Since the Christoffel symbols are homogeneous and linear in the first derivatives of the metric tensor, and since the $\eta_{\alpha\beta}$ have vanishing derivatives, each of the first two terms of (9.3) will contain a single factor of the parameter ϵ. The second two terms, however, are products of Christoffel symbols and therefore will contain factors of ϵ^2; by our approximation scheme, these second two terms are to be ignored. Equation (9.3) to first order in ϵ is then

$$\left\{ {\beta \atop \beta\ \eta} \right\}_{|\lambda} - \left\{ {\beta \atop \lambda\ \eta} \right\}_{|\beta} = 0 \tag{9.4}$$

The first contracted Christoffel symbol may be simplified; from (3.11) we have

$$\left\{ {\beta \atop \beta\ \eta} \right\} = \tfrac{1}{2}(\log |g|)_{|\eta} \tag{9.5}$$

so we can put (9.4) in the form

$$\tfrac{1}{2}(\log |g|)_{|\eta|\lambda} - \left\{ {\beta \atop \lambda\ \eta} \right\}_{|\beta} = 0 \tag{9.6}$$

The function $|g|$, the absolute value of the metric-tensor determinant, can be written, using (9.1), as

$$|g| = \|-g_{\alpha\beta}\| = \begin{Vmatrix} 1-\epsilon\gamma_{00} & -\epsilon\gamma_{01} & -\epsilon\gamma_{02} & -\epsilon\gamma_{03} \\ -\epsilon\gamma_{10} & 1-\epsilon\gamma_{11} & -\epsilon\gamma_{12} & -\epsilon\gamma_{13} \\ -\epsilon\gamma_{20} & -\epsilon\gamma_{21} & 1-\epsilon\gamma_{22} & -\epsilon\gamma_{23} \\ -\epsilon\gamma_{30} & -\epsilon\gamma_{31} & -\epsilon\gamma_{32} & 1-\epsilon\gamma_{33} \end{Vmatrix} \tag{9.7}$$

But the only term of first order in ϵ which occurs in this determinant comes from the product of the diagonal elements, so to first order in ϵ, we have

$$\begin{aligned} |g| &= (1-\epsilon\gamma_{00})(1-\epsilon\gamma_{11})(1-\epsilon\gamma_{22})(1-\epsilon\gamma_{33}) \\ &= 1-\epsilon(\gamma_{00}+\gamma_{11}+\gamma_{22}+\gamma_{33}) \\ &= 1-\epsilon\,\mathrm{Tr}\,\gamma \end{aligned} \tag{9.8}$$

Expanding $\log |g|$ in a Taylor series to first order in ϵ, we obtain

$$\log |g| = \log (1-\epsilon\,\mathrm{Tr}\,\gamma) = -\epsilon\,\mathrm{Tr}\,\gamma \tag{9.9}$$

Thus the first term of the linearized equations (9.6) may be written as

$$\tfrac{1}{2}(\log |g|)_{|\eta|\gamma} = -\tfrac{1}{2}\epsilon(\mathrm{Tr}\,\gamma)_{|\eta|\gamma} = -\tfrac{1}{2}\epsilon \sum_{\beta=0}^{3} \gamma_{\beta\beta|\eta|\gamma} \tag{9.10}$$

The second Christoffel symbol which appears in (9.4) may be written out as

$$\left\{ {\beta \atop \lambda\ \eta} \right\} = \frac{g^{\beta\mu}}{2}[\lambda\eta,\mu] = \frac{\eta^{\beta\mu}+\epsilon\gamma^{\beta\mu}}{2}(\epsilon\gamma_{\mu\lambda|\eta}+\epsilon\gamma_{\mu\eta|\lambda}-\epsilon\gamma_{\lambda\eta|\mu}) \tag{9.11}$$

Using the explicit form given for $\eta_{\alpha\beta}$ in (9.2), we can write this to first order in ϵ as

$$\left\{ {\beta \atop \lambda\ \eta} \right\} = -\tfrac{1}{2}(\epsilon\gamma_{\beta\lambda|\eta}+\epsilon\gamma_{\beta\eta|\lambda}-\epsilon\gamma_{\lambda\eta|\beta}) \tag{9.12}$$

(Note that we are *not* raising or lowering indices in the above.) Hence, to first order in ϵ, we obtain

$$\left\{ {\beta \atop \lambda\ \eta} \right\}_{|\beta} = -\sum_{\beta=0}^{3} \tfrac{1}{2}(\epsilon\gamma_{\beta\lambda|\eta}+\epsilon\gamma_{\beta\eta|\lambda}-\epsilon\gamma_{\lambda\eta|\beta})_{|\beta} \tag{9.13}$$

By use of (9.10) and (9.13), the linearized equations (9.3) can now be written completely in terms of the perturbation $\gamma_{\alpha\beta}$ as

$$\sum_{\beta=0}^{3} \gamma_{\beta\beta|\eta|\lambda} - \sum_{\beta=0}^{3} (\gamma_{\beta\eta|\lambda|\beta} + \gamma_{\lambda\beta|\eta|\beta} - \gamma_{\eta\lambda|\beta|\beta}) = 0 \tag{9.14}$$

We have now obtained a system of 10 partial differential equations for the 10 unknown components of the symmetric $\gamma_{\alpha\beta}$ perturbation term. In the following sections we shall investigate and simplify these equations.

We should note at this point that, since we delete terms of second and higher order in the parameter ϵ, we no longer have a covariant theory; that is, the linearized equations are not covariant and the solution $\gamma_{\alpha\beta}$ is therefore not necessarily a tensor. However, it is easily shown that we may treat the approximate metric tensor $g_{\alpha\beta}^{(A)} = \eta_{\alpha\beta} + \epsilon\gamma_{\alpha\beta}$ as a true tensor under coordinate transformations consistently to order ϵ. To show this, let us form the difference between the exact metric tensor $g_{\alpha\beta}$, which is a solution of the complete field equations (9.3), and the corresponding approximate solution $g_{\alpha\beta}^{(A)}$ obtained from the linearized equations (9.14). According to our approximation scheme, this difference is of order ϵ^2. Thus we may write

$$g_{\alpha\beta} - g_{\alpha\beta}^{(A)} = O(\epsilon^2) \tag{9.15}$$

Let us now tentatively treat $g_{\alpha\beta}^{(A)}$ as a tensor and define in a barred coordinate system, which may or may not be Minkowskian,

$$\bar{g}_{\mu\nu}^{(A)} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} g_{\alpha\beta}^{(A)} \tag{9.16}$$

It follows that the difference between the "approximate metric tensor" so defined and the exact metric tensor in the new system is

$$\bar{g}_{\mu\nu} - \bar{g}_{\mu\nu}^{(A)} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} (g_{\alpha\beta} - g_{\alpha\beta}^{(A)}) = O(\epsilon^2) \tag{9.17}$$

Thus, by treating $g_{\alpha\beta}^{(A)}$ as a true tensor under coordinate transformation, we make an error at most of order ϵ^2. Since we shall carry out all our calculations with deliberate neglect of all terms containing the factor ϵ^2, we may treat $g_{\alpha\beta}^{(A)}$ consistently as an ordinary tensor. Furthermore, it is clear that, if the above transformation is to a new Minkowski frame where we can solve the linearized equations, then the approximate metric defined by (9.16) and the approximate metric obtained by solving the linearized equations in the new frame can differ at most by order ϵ^2.

9.2 The Time-independent and Spherically Symmetric Field

We possess in the linearized field equations (9.14) a convenient tool for studying the influence of weak perturbations on a flat-space metric. Clearly, the use of linear equations for the calculation of the metric tensor is closest to the approach of classical potential theory and allows the best comparison between classical and relativistic theory. Historically, the linearized theory was first used to investigate the static gravitational field with spherical symmetry; the experimentally detectable effects which we studied in Secs. 4.4, 6.3, and 6.5 were first obtained by Einstein in this manner in 1915. Shortly afterward, in 1916, Schwarzschild succeeded in obtaining an exact solution to the same problem, which we have already studied in Chap. 6. We shall now study the approximate solution of Einstein. The reason for doing so is twofold. First, we present Einstein's argument for historical interest, and second, we obtain experience in handling the system (9.14). It will become apparent that even in cases where an exact solution can be obtained, the linearized solution is valuable, in that it is more accessible to calculation and is often more open to physical interpretation.

As we discussed in Chap. 6, the meaning of a static gravitational field may be summed up in the form of its line element

$$ds^2 = g_{00}(dx^0)^2 + g_{ik}\,dx^i\,dx^k \tag{9.18}$$

where g_{00} and g_{ik} depend only on the space variables x^i. If we use Minkowski coordinates as before, we have

$$x_0 = ict \tag{9.19}$$

and hence we can express (9.18) as

$$ds^2 = -g_{00}c^2\,dt^2 + g_{ik}\,dx^i\,dx^k \tag{9.20}$$

with a negative definite g_{ik} matrix.

In order to carry out the linearization described in the preceding section, we introduce a function $a(x^i)$ defined by

$$g_{00} = -1 + \epsilon a \qquad a = \gamma_{00} \qquad \gamma_{0i} = 0 \tag{9.21}$$

and write g_{ik} in the form

$$g_{ik} = -\delta_{ik} + \epsilon\gamma_{ik} \tag{9.22}$$

With these definitions we can write the $\eta = \lambda = 0$ component of the linearized system (9.14),

$$\sum_{\beta=0}^{3} \gamma_{\beta\beta|0|0} - \sum_{\beta=0}^{3} (\gamma_{\beta0|0|\beta} + \gamma_{0\beta|0|\beta} - \gamma_{00|\beta|\beta}) = 0 \tag{9.23}$$

in a simple form. Since all $\gamma_{\alpha\beta}$ are independent of x^0, the only surviving term of (9.23) is

$$\sum_{\beta=0}^{3} \gamma_{00|\beta|\beta} = \sum_{k=1}^{3} a_{|k|k} = 0 \tag{9.24}$$

In terms of the Laplacian operator ∇^2, this is simply the harmonic equation in three dimensions,

$$\nabla^2 a = \nabla^2 \gamma_{00} = 0 \tag{9.25}$$

Thus we have shown that γ_{00} of the perturbation matrix satisfies Laplace's equation and is therefore a harmonic function.

Let us digress for a moment and consider the implications of (9.25) with regard to the correspondence between general relativity theory and classical gravitational theory. In studying the geodesic equations of motion in Sec. 4.3, we found that they correspond to Newton's classical equations of motion in a potential field in the limit of low velocities and weak fields. However, in order to make the correspondence hold, we had to assume an asymptotic relation between the g_{00} component of the metric tensor and the classical potential φ, which describes the gravitational field; in terms of the coordinates of special relativity ct, x, y, and z, that relation was

$$g_{00} = 1 + \frac{2\varphi}{c^2} \qquad \text{(coordinates of special relativity)} \tag{9.26}$$

[Eq. (4.142)]. In terms of the Minkowski coordinates ict, x, y, and z, which we are now using, (9.26) becomes

$$g_{00} = -1 - \frac{2\varphi}{c^2} \qquad \text{(Minkowski coordinates)} \tag{9.27}$$

Using the above equation and the definition of the perturbation matrix $\gamma_{\mu\nu}$ in (9.1), we see that we must have approximately

$$\epsilon\gamma_{00} = \epsilon a = -\frac{2\varphi}{c^2} \tag{9.28}$$

This equation allows us to deduce a field equation for the classical potential φ on the basis of relativity theory. Indeed, from (9.25), we obtain

$$\nabla^2\varphi = 0 \tag{9.29}$$

which is the same equation, Laplace's equation, that φ satisfies according to classical theory.

Let us note that the identification of geodesic motion in relativity theory with Newtonian motion in a potential field led to the correspondence (9.26) between metric and potential functions without the use of Einstein's field equations. We now see that Einstein's field equations in the linearized approximation are consistent with the correspondence, and furthermore reduce precisely to the correct classical equation in the limit considered above.

One other fact concerning the classical correspondence of general relativity is worthy of note at this point. It is well known in classical potential theory that if the harmonic function φ satisfies the boundary condition that it be zero at infinity, then it must be zero everywhere, unless Laplace's equation breaks down at some point in space or in some extended region of space. Physically, this means that there must be some point (a particle) or some region (an extended body) where $\nabla^2\varphi$ is nonzero, or else there can be no gravitational field. Clearly, the same result holds for the function $\gamma_{00} = a$ by virtue of Eq. (9.28).

The above result is analogous to the uniqueness theorem of Sec. 8.6. However, it is not a special case of that theorem, since we are now dealing with the linearized theory only.

Until now we have used only the assumption that the gravitational field is static. Let us now add the assumption of radial symmetry. As we discussed in Sec. 6.2, a radially symmetric static line element can be put into the isotropic form

$$ds^2 = -g_{00}c^2\,dt^2 - g_{11}(dx^2 + dy^2 + dz^2) \tag{9.30}$$

If we denote the usual radial distance by $r = \sqrt{x^2 + y^2 + z^2}$, we can, moreover, assert that, because of radial symmetry,

$$g_{00} = -1 + \epsilon a(r) \qquad g_{11} = -1 + \epsilon b(r) \tag{9.31}$$

and thus the perturbation matrix $\gamma_{\alpha\beta}$ assumes the simple form

$$\gamma_{\alpha\beta}(r) = \begin{pmatrix} a(r) & 0 & 0 & 0 \\ 0 & b(r) & 0 & 0 \\ 0 & 0 & b(r) & 0 \\ 0 & 0 & 0 & b(r) \end{pmatrix} \tag{9.32}$$

By the correspondence (9.28) we can easily obtain the function $a(r)$ explicitly; indeed, the classical potential of a spherically symmetric field (with a singularity at $r = 0$) is simply $\varphi = -\kappa M/r$, where M is the mass of the body at $r = 0$ and κ is the gravitational constant. Thus, from (9.28),

(9.33) $$\epsilon a(r) = \frac{2\kappa M}{c^2 r}$$

We next wish to determine the remaining unknown function $b(r)$ by investigating the remaining linearized field equations for $\gamma_{11} = \gamma_{22} = \gamma_{33}$. We set $\eta = \lambda = j$ in the linearized equations (9.14) to obtain

(9.34) $$\sum_{\beta=0}^{3} \gamma_{\beta\beta|j|j} = \sum_{\beta=0}^{3} (\gamma_{\beta j|j|\beta} + \gamma_{j\beta|j|\beta} - \gamma_{jj|\beta|\beta})$$

In terms of $a(r)$ and $b(r)$, the left side of (9.34) becomes

(9.35) $$\sum_{\beta=0}^{3} \gamma_{\beta\beta|j|j} = \gamma_{00|j|j} + \sum_{k=1}^{3} \gamma_{kk|j|j} = a_{|j|j} + 3b_{|j|j}$$

Since the matrix $\gamma_{\alpha\beta}$ is diagonal, symmetric in the indices α and β, and independent of time, the right side of (9.34) becomes

(9.36) $$\sum_{\beta=0}^{3} (\gamma_{\beta j|j|\beta} + \gamma_{j\beta|j|\beta} - \gamma_{jj|\beta|\beta}) = 2\gamma_{jj|j|j} - \sum_{k=1}^{3} \gamma_{jj|k|k}$$
$$= 2b_{|j|j} - \sum_{k=1}^{3} b_{|k|k}$$

Using Eqs. (9.35) and (9.36), we can rewrite the linearized equation (9.34) in the form

(9.37) $$a_{|j|j} + 3b_{|j|j} = 2b_{|j|j} - \sum_{k=1}^{3} b_{|k|k}$$

In terms of the Laplacian operator ∇^2, this can be written as

(9.38) $$a_{|j|j} + b_{|j|j} + \nabla^2 b = 0$$

Now let us sum Eqs. (9.38) over values of j and recall that $\nabla^2 a = 0$. We find

(9.39) $$\nabla^2 b = 0$$

and hence (9.38) reduces to

(9.40) $$(a + b)_{|j|j} = 0$$

We conclude that $a + b$ is a linear function of the coordinates, and since it is zero at infinity, it is zero everywhere. Hence we have proved $a = -b$, and we therefore arrive at the following perturbation matrix:

(9.41) $$\gamma_{\alpha\beta} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}$$

The approximate metric tensor in Minkowski coordinates follows from (9.41):

(9.42) $$g_{\alpha\beta} = \begin{pmatrix} -1+\epsilon a & 0 & 0 & 0 \\ 0 & -1-\epsilon a & 0 & 0 \\ 0 & 0 & -1-\epsilon a & 0 \\ 0 & 0 & 0 & -1-\epsilon a \end{pmatrix}$$

The line element is, accordingly,

(9.43) $$ds^2 = (1 - \epsilon a)c^2\, dt^2 - (1 + \epsilon a)(dx^2 + dy^2 + dz^2)$$

Using the classical correspondence relations (9.28) and (9.33), we can write this in the form

(9.44) $$ds^2 = \left(1 + \frac{2\varphi}{c^2}\right) c^2\, dt^2 - \left(1 - \frac{2\varphi}{c^2}\right)(dx^2 + dy^2 + dz^2)$$
$$= \left(1 - \frac{2kM}{c^2 r}\right) c^2\, dt^2 - \left(1 + \frac{2kM}{c^2 r}\right) d\sigma^2$$

where $d\sigma^2 = dx^2 + dy^2 + dz^2$. This result agrees with the exact Schwarzschild solution in isotropic coordinates, which we studied in Chap. 6.

9.3 The Weyl Solutions to the Linearized Field Equations

In this section we shall obtain a particularly interesting class of solutions to the linearized equations which is due to Weyl (Weyl, 1918). The rest of the chapter will then be devoted to investigating the relation of the Weyl solutions to the structure of the linearized equations.

We return now to the general linearized equations (9.14) and define the four-dimensional D'Alembertian operator

$$\Box^2\gamma_{\eta\lambda} = -\sum_{\beta=0}^{3} \gamma_{\eta\lambda|\beta|\beta} \tag{9.45}$$

In terms of this operator the linearized field equations (9.14) can be written as

$$\Box^2\gamma_{\eta\lambda} + \sum_{\beta=0}^{3} \gamma_{\lambda\beta|\eta|\beta} + \sum_{\beta=0}^{3} \gamma_{\beta\eta|\lambda|\beta} - \sum_{\beta=0}^{3} \gamma_{\beta\beta|\eta|\lambda} = 0 \tag{9.46}$$

which can be rearranged into the symmetric form

$$\Box^2\gamma_{\eta\lambda} + \sum_{\beta=0}^{3} (\gamma_{\lambda\beta|\beta} - \tfrac{1}{2}\gamma_{\beta\beta|\lambda})_{|\eta} + \sum_{\beta=0}^{3} (\gamma_{\eta\beta|\beta} - \tfrac{1}{2}\gamma_{\beta\beta|\eta})_{|\lambda} = 0 \tag{9.46$'$}$$

If we define a four-component quantity τ_η as

$$\tau_\eta = -\sum_{\beta=0}^{3} (\gamma_{\eta\beta|\beta} - \tfrac{1}{2}\gamma_{\beta\beta|\eta}) \tag{9.47}$$

this can be written in more compact form as

$$\Box^2\gamma_{\eta\lambda} = \tau_{\lambda|\eta} + \tau_{\eta|\lambda} \tag{9.48}$$

which is completely equivalent to (9.14).

Following Weyl, let us now tentatively set τ_λ equal to the D'Alembertian of a four-component function φ_λ,

$$\tau_\lambda = \Box^2\varphi_\lambda \tag{9.49}$$

which can be obtained by solving the inhomogeneous wave equation. We substitute this in (9.48):

$$\Box^2\gamma_{\eta\lambda} = \Box^2\varphi_{\lambda|\eta} + \Box^2\varphi_{\eta|\lambda} \tag{9.50}$$

This leads us to investigate the possibility that a solution for the matrix $\gamma_{\eta\lambda}$ might be

$$\gamma_{\eta\lambda} = \varphi_{\lambda|\eta} + \varphi_{\eta|\lambda}, \tag{9.51}$$

where the φ_λ are four arbitrarily assigned functions. By substituting this back into (9.46), we can easily verify that it is indeed a solution. The left side of (9.46) becomes

$$\begin{aligned} \Box^2\gamma_{\eta\lambda} &+ \sum_{\beta=0}^{3} (\gamma_{\lambda\beta|\eta|\beta} + \gamma_{\beta\eta|\lambda|\beta} - \gamma_{\beta\beta|\eta|\lambda}) \\ &= \Box^2\gamma_{\eta\lambda} + \sum_{\beta=0}^{3} (\varphi_{\lambda|\beta|\eta|\beta} + \varphi_{\beta|\lambda|\eta|\beta} + \varphi_{\beta|\eta|\lambda|\beta} + \varphi_{\eta|\beta|\lambda|\beta} - \varphi_{\beta|\beta|\eta|\lambda} - \varphi_{\beta|\beta|\eta|\lambda}) \\ &= \Box^2\gamma_{\eta\lambda} - (\Box^2\varphi_{\lambda|\eta} + \Box^2\varphi_{\eta|\lambda}) \end{aligned} \tag{9.52}$$

By virtue of (9.50) this is zero, so the expression (9.51) is indeed a solution of the field equations.

Let us consider for a moment the above result. We began with the general linearized system (9.46) of 10 equations for the 10 independent elements of the symmetric matrix $\gamma_{\eta\lambda}$. By Weyl's *Ansatz* (9.51) we were able to generate a large subclass of solutions by using an arbitrary twice-differentiable set of four functions φ_λ. Solutions which belong to this subclass, i.e., have the form $\varphi_{\lambda|\eta} + \varphi_{\eta|\lambda}$, are termed Weyl solutions. We shall see that they form a very important subclass of solutions to the linearized equations.

Consider now an arbitrary solution of the linearized equations $\gamma_{\eta\lambda}$. Using this solution, we define a set of four associated τ_η functions as

$$\tau_\eta = -\sum_{\beta=0}^{3} (\gamma_{\eta\beta|\beta} - \tfrac{1}{2}\gamma_{\beta\beta|\eta}) \tag{9.53}$$

We can also define a set of associated φ_λ functions by the four equations

$$\Box^2\varphi_\lambda = \tau_\lambda \tag{9.54}$$

The resultant set of functions φ_λ, which we have generated from the original solution of the field equations $\gamma_{\eta\lambda}$, can now serve as a generating function for a new solution of the field equations of the Weyl form

$$\gamma_{\eta\lambda}^{(w)} = \varphi_{\eta|\lambda} + \varphi_{\lambda|\eta} \tag{9.55}$$

which, we note, has the same set of associated τ_η functions (9.53) as $\gamma_{\eta\lambda}$. This solution which we have generated will be termed the associated Weyl solution of $\gamma_{\eta\lambda}$. Thus, to every solution of the linearized equations, there corresponds a unique *associated* Weyl solution.

There is also another type of solution associated with the arbitrary solution $\gamma_{\eta\lambda}$ which will prove to be of interest; consider the difference between $\gamma_{\eta\lambda}$ and its associated Weyl solution $\gamma_{\eta\lambda}^{(w)}$:

$$\gamma_{\eta\lambda} - \gamma_{\eta\lambda}^{(w)} = \hat{\gamma}_{\eta\lambda} \tag{9.56}$$

Since the equations (9.14) are linear, this is indeed a solution, and we note, furthermore, that it is uniquely determined by $\gamma_{\eta\lambda}$.

Let us investigate this solution by first computing its D'Alembertian. From the definition of τ_η in (9.53) and the field equations in the form (9.48), the D'Alembertian of $\gamma_{\eta\lambda}$ is simply

$$\Box^2\gamma_{\eta\lambda} = \tau_{\eta|\lambda} + \tau_{\lambda|\eta} \tag{9.57}$$

Similarly, from the definitions of $\gamma_{\eta\lambda}^{(w)}$ in (9.55) and of φ_λ in (9.54), we obtain

$$\Box^2\gamma_{\eta\lambda}^{(w)} = \Box^2\varphi_{\eta|\lambda} + \Box^2\varphi_{\lambda|\eta} = \tau_{\eta|\lambda} + \tau_{\lambda|\eta} \tag{9.58}$$

Thus the solution $\gamma_{\eta\lambda}$ and its associated Weyl solution have the same D'Alembertian. The difference of these solutions $\hat{\gamma}_{\eta\lambda}$ therefore has a null D'Alembertian:

$$\Box^2\hat{\gamma}_{\eta\lambda} = 0 \tag{9.59}$$

This is a very important result, for Eq. (9.59) is the familiar wave equation of classical physics. It states that the 10-component disturbance which is represented by $\hat{\gamma}_{\eta\lambda}$ is *propagated with velocity c, the speed of light.*

We shall next obtain a set of equations which relates the components $\hat{\gamma}_{\eta\lambda}$ to each other. The set of functions $\hat{\tau}_\eta$ associated with $\hat{\gamma}_{\eta\lambda}$ and defined by an equation analogous to (9.53) obeys the equations

$$\Box^2\hat{\gamma}_{\eta\lambda} = \hat{\tau}_{\eta|\lambda} + \hat{\tau}_{\lambda|\eta} = 0 \tag{9.60}$$

Vector fields ξ_λ on Riemannian manifolds with the differential condition

$$\xi_{\lambda\|\eta} + \xi_{\eta\|\lambda} = 0 \tag{9.61}$$

have been discussed in Sec. 3.7. We recall that they are called Killing vector fields and indicate that a symmetry of the metric is present. In our present approximation (9.61) implies that $\hat{\tau}_\eta$ is a field of Killing vectors. We shall now show that a field of such vectors which is regular everywhere and vanishes at infinity in a space which is asymptotically pseudo-Euclidean is identically zero. By asymptotically pseudo-Euclidean, we mean here that, as the *space* coordinates x^i go to infinity, the metric becomes asymptotically pseudo-Euclidean.

Differentiation of (9.60) with respect to x^ν gives

$$\hat{\tau}_{\eta|\lambda|\nu} + \hat{\tau}_{\lambda|\eta|\nu} = 0 \tag{9.62}$$

By cyclic permutation of the indices η, λ, and ν, we then obtain the system of equations

$$\begin{aligned} \hat{\tau}_{\eta|\lambda|\nu} + \hat{\tau}_{\lambda|\nu|\eta} &= 0 \\ \hat{\tau}_{\lambda|\nu|\eta} + \hat{\tau}_{\nu|\eta|\lambda} &= 0 \\ \hat{\tau}_{\nu|\eta|\lambda} + \hat{\tau}_{\eta|\lambda|\nu} &= 0 \end{aligned} \tag{9.63}$$

These may conveniently be written in matrix form as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\tau}_{\eta|\lambda|\nu} \\ \hat{\tau}_{\lambda|\eta|\nu} \\ \hat{\tau}_{\nu|\lambda|\eta} \end{pmatrix} = 0 \tag{9.63'}$$

since the indices of ordinary differentiation commute. This system can have a nonzero solution only if the determinant of the coefficient matrix is zero, but the determinant is clearly equal to 2; thus the only solution of the system is the null solution

$$\hat{\tau}_{\eta|\lambda|\nu} = 0 \tag{9.64}$$

By integration, we then have

$$\hat{\tau}_\eta = \text{linear function} \tag{9.65}$$

However, a linear function is either zero, constant, or infinite for large arguments. For most reasonable physical systems (such as isolated bodies or gravitational waves), the gravitational field should be asymptotically zero in at least one spatial direction, as we have already assumed. This requires that $\gamma_{\alpha\beta}$ behave similarly and that the linear function in (9.65) be identically zero; that is,

$$\hat{\tau}_\eta = 0 \tag{9.66}$$

Substituting the definition of $\hat{\tau}_\eta$ in (9.66), we have, finally,

$$\hat{\tau}_\eta = \sum_{\beta=0}^{3} (\hat{\gamma}_{\eta\beta|\beta} - \tfrac{1}{2}\hat{\gamma}_{\beta\beta|\eta}) = 0 \tag{9.67}$$

which is the set of relations on the components of $\hat{\gamma}_{\eta\lambda}$ that we desired.

We shall see in the next section that the matrices $\hat{\gamma}_{\eta\lambda}$ associated with arbitrary solutions $\gamma_{\eta\lambda}$ are indeed physically more important and meaningful than the original arbitrary solutions.

9.4 Structure of the Linearized Equations

In the preceding section we found that, with each solution of the linearized field equations $\gamma_{\eta\lambda}$, we may associate two other solutions: one is the associated Weyl solution $\gamma_{\eta\lambda}^{(w)}$, and the other is the difference between the original solution and its associated Weyl solution $\gamma_{\eta\lambda} - \gamma_{\eta\lambda}^{(w)}$, which we call $\hat{\gamma}_{\eta\lambda}$. Now we wish to show that the only solution with physical importance is the second associated solution $\hat{\gamma}_{\eta\lambda}$.

Recall from Chap. 5 that the Riemann tensor $R^{\alpha}{}_{\eta\beta\lambda}$ determines several important properties of space; in particular, a null curvature tensor is a necessary and sufficient condition that a space be Lorentzian or pseudo-Euclidean (Sec. 5.6). Therefore, as a first step in investigating the role played by the Weyl solutions in the structure of the linearized equations, we shall calculate the Riemann tensor for a metric tensor of the form $g_{\alpha\beta}^{(L)} + \epsilon\gamma_{\alpha\beta}^{(w)}$. By definition (Sec. 5.2), the Riemann tensor is

$$(9.68)\qquad R^{\alpha}{}_{\eta\beta\lambda} = \left\{ {\alpha \atop \beta\ \eta} \right\}_{|\lambda} - \left\{ {\alpha \atop \eta\ \lambda} \right\}_{|\beta} + \left\{ {\alpha \atop \tau\ \lambda} \right\}\left\{ {\tau \atop \beta\ \eta} \right\} - \left\{ {\alpha \atop \tau\ \beta} \right\}\left\{ {\tau \atop \lambda\ \eta} \right\}$$

Because of the form of the metric tensor, each Christoffel symbol is of order ϵ, so the last two terms of the Riemann tensor are of order ϵ^2 and may be deleted in the linearized theory. This leaves

$$(9.69)\qquad R^{\alpha}{}_{\eta\beta\lambda} = \left\{ {\alpha \atop \beta\ \eta} \right\}_{|\lambda} - \left\{ {\alpha \atop \eta\ \lambda} \right\}_{|\beta}$$

The Christoffel symbols have already been investigated in Sec. 9.1. To first order in ϵ we may write the Christoffel symbol as

$$(9.70)\qquad \left\{ {\alpha \atop \beta\ \eta} \right\} = -\frac{\epsilon}{2}(\gamma_{\alpha\beta|\eta} + \gamma_{\alpha\eta|\beta} - \gamma_{\beta\eta|\alpha})$$

Substitution of this in (9.69) gives

$$(9.71)\qquad R^{\alpha}{}_{\eta\beta\lambda} = \frac{\epsilon}{2}(\gamma_{\alpha\lambda|\eta|\beta} + \gamma_{\beta\eta|\alpha|\lambda} - \gamma_{\alpha\beta|\eta|\lambda} - \gamma_{\eta\lambda|\alpha|\beta})$$

This equation is valid for an arbitrary perturbation matrix $\gamma_{\alpha\beta}$. For the special case of a Weyl solution the matrix $\gamma_{\alpha\beta}$ has the form $\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}$.

The corresponding Riemann tensor is therefore

$$(9.72)\qquad \begin{aligned} R^{\alpha}{}_{\eta\beta\lambda} &= \frac{\epsilon}{2}(\varphi_{\alpha|\lambda|\eta|\beta} + \varphi_{\lambda|\alpha|\eta|\beta} + \varphi_{\beta|\eta|\alpha|\lambda} + \varphi_{\eta|\beta|\alpha|\lambda} \\ &\qquad - \varphi_{\alpha|\beta|\eta|\lambda} - \varphi_{\beta|\alpha|\eta|\lambda} - \varphi_{\eta|\lambda|\alpha|\beta} - \varphi_{\lambda|\eta|\alpha|\beta}) \\ &= 0 \end{aligned}$$

The Weyl solution gives rise to a null Riemann tensor (to first order in ϵ).

In the case of a more general solution $\gamma_{\alpha\beta}$, we may use the decomposition $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}^{(w)} + \hat{\gamma}_{\alpha\beta}$. From (8.71) it is clear that, within our approximation, the Riemann tensor depends linearly on $\gamma_{\alpha\beta}$ and can be written as the sum of a term which depends only on $\gamma_{\alpha\beta}^{(w)}$ and a term which depends only on $\hat{\gamma}_{\alpha\beta}$; we have shown above that the term depending on $\gamma_{\alpha\beta}^{(w)}$ is identically zero, so the entire Riemann tensor depends only on $\hat{\gamma}_{\alpha\beta}$. Since the Weyl term is irrelevant in determining the curvature tensor of the Riemann space, we are led to suspect that it corresponds to a *formal* property of the linearized equations and is of no physical consequence. We shall show below that this is indeed the case; the Weyl solution stems entirely from the freedom we have in choosing a Minkowski coordinate system and can furthermore be eliminated by an appropriate choice of coordinate system.

Our assumption on the form of the metric tensor

$$(9.73)\qquad g_{\alpha\beta} = -\delta_{\alpha\beta} + \epsilon\gamma_{\alpha\beta}$$

allows a coordinate transformation of the form

$$(9.74)\qquad \bar{x}^{\mu} = x^{\mu} - \epsilon\varphi_{\mu}$$

which does not change the form of the metric (9.73). Let us show explicitly the effect of (9.74) on the metric tensor; the transformation coefficients to first order in ϵ are

$$(9.75)\qquad \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} = \delta^{\mu}{}_{\alpha} - \epsilon\frac{\partial \varphi_{\mu}}{\partial x^{\alpha}} \qquad \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} = \delta^{\alpha}{}_{\mu} + \epsilon\frac{\partial \varphi_{\alpha}}{\partial \bar{x}^{\mu}}$$

Thus, in the new system, the metric tensor is

$$(9.76)\qquad \begin{aligned} \bar{g}_{\mu\nu} &= \left(\delta^{\alpha}{}_{\mu} + \epsilon\frac{\partial\varphi_{\alpha}}{\partial\bar{x}^{\mu}}\right)\left(\delta^{\beta}{}_{\nu} + \epsilon\frac{\partial\varphi_{\beta}}{\partial\bar{x}^{\nu}}\right)(-\delta_{\alpha\beta} + \epsilon\gamma_{\alpha\beta}) \\ &= -\delta_{\mu\nu} + \epsilon\left[\gamma_{\mu\nu} - \frac{\partial\varphi_{\nu}}{\partial\bar{x}^{\mu}} - \frac{\partial\varphi_{\mu}}{\partial\bar{x}^{\nu}}\right] \\ &= -\delta_{\mu\nu} + \epsilon\,[\gamma_{\mu\nu} - \varphi_{\nu|\mu} - \varphi_{\mu|\nu}] \end{aligned}$$

Thus the metric tensor in the new system has the same form as in (9.73), but we see that a Weyl-type solution, $-(\varphi_{\nu|\mu} + \varphi_{\mu|\nu})$, has been added to $\gamma_{\alpha\beta}$. Hence the influence of the arbitrariness of the coordinate system due to the freedom of choosing the function φ_μ in (9.74) is to add a Weyl-type solution to $\gamma_{\alpha\beta}$.

By a proper choice of the functions φ_μ used in the above discussion, it is clear that the Weyl part of any solution $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}^{(w)} + \hat{\gamma}_{\alpha\beta}$ can be eliminated by the simple coordinate transformation (9.74). Moreover, if $\gamma_{\alpha\beta}$ were entirely a Weyl type of solution, we could easily go to a system where $\gamma_{\alpha\beta} = -\delta_{\alpha\beta}$, which would imply a pseudo-Euclidean space. This agrees with the fact that the Riemann tensor corresponding to a Weyl solution is identically zero, for, as we discussed in Chap. 5, a space with a null curvature tensor is pseudo-Euclidean, and vice versa.

Let us consider for a moment our results and their significance in the structure of the linearized equations. We found that the presence of the Weyl solution in the metric tensor $g_{\alpha\beta}^{(L)} + \gamma_{\alpha\beta}^{(w)}$ gave rise to a null Riemann tensor. Therefore the Weyl solution cannot correspond to a gravitational field. Indeed, we have seen that the transformation (9.73) can explicitly remove the Weyl solution from the metric tensor. We must therefore consider the Weyl solution to be a formal property of the coordinate system we are using, and not a physical property of space. A problem, therefore, confronts us: How can we separate out the Weyl solutions so that, in general, we deal only with physically meaningful solutions? One method of consistently discarding the Weyl solutions would be to obtain first a solution $\gamma_{\alpha\beta}$, then compute its associated Weyl solution $\gamma_{\alpha\beta}^{(w)}$ using (9.53) and (9.54), and then subtract $\gamma_{\alpha\beta}^{(w)}$ from $\gamma_{\alpha\beta}$ to form $\hat{\gamma}_{\alpha\beta}$, which is the physically meaningful solution. However, there is a much simpler course open to us whereby the Weyl solutions are automatically excluded from consideration and never appear in our calculations at all. We found in the preceding sections that the solution $\hat{\gamma}_{\alpha\beta}$ alone obeys several sets of equations—the second-order D'Alembertian differential equation

(9.77) $$\square^2 \hat{\gamma}_{\alpha\beta} = 0$$

and a set of first-order differential equations relating the components

(9.78) $$\sum_{\beta=0}^{3} (\hat{\gamma}_{\eta\beta|\beta} - \tfrac{1}{2}\hat{\gamma}_{\beta\beta|\eta}) = c_\eta$$

in which the constant c_η is zero for Euclidean boundary conditions at infinity (as we discussed in Sec. 9.3) and depends on the specific boundary conditions otherwise. These two sets of equations form a very elegant restatement of the linearized equations since they are indeed equivalent to the linearized equations with the Weyl solution automatically discarded.

Let us note, finally, that one of the most important results of the linearized theory is the fact that the physically interesting solutions $\hat{\gamma}_{\alpha\beta}$ of the linearized equations satisfy the wave equation (9.77). Hence perturbations in the metric field (gravitational "waves") satisfy the same differential equation as electromagnetic phenomena. However, the 10 components of the metric field $\hat{\gamma}_{\alpha\beta}$ are also coupled by the four relations (9.78), which implies that properties such as the polarization configurations of the metric field may differ considerably from the electromagnetic field. The fact that electromagnetic and gravitational disturbances follow the same paths (null geodesics) was shown in general, in Chap. 7, with no approximations. However, the general approach did not reveal the four relations (9.78) which couple the components of the metric field. Thus both the general approach and the linearized approach to the theory of gravitational waves have certain advantages and certain drawbacks; the use of both together in the study of gravitational propagation is clearly of value.

9.5 Gravitational Waves

A great deal of thought has been devoted in recent years to the theoretical analysis of gravitational waves, but the detection of such waves is a very difficult experimental task only recently attempted. We shall say more about these efforts at detection later in this section. For clarity our brief theoretical analysis of gravitational waves will deal only with plane-wave solutions to the linearized field equations.

Let us begin by asking what theoretical motivation there is to believe that gravitational waves exist. The linearized equations may be written in the concise form

(9.79*a*) $$\square^2 \gamma_{\mu\nu} = 0$$

(9.79*b*) $$\sum_{\beta=0}^{3} (\gamma_{\eta\beta|\beta} - \tfrac{1}{2}\gamma_{\beta\beta|\eta}) = 0$$

as we showed in the preceding section. We have now dropped the $\hat{\gamma}_{\mu\nu}$ notation for convenience, and we shall assume Euclidean boundary conditions at infinity. One might be tempted to interpret (9.79*a*) as indicating that gravitational effects propagate as waves with velocity c. This, however, is open to the objection that the perturbation $\gamma_{\mu\nu}$ is linked to an arbitrary coordinate system, and therefore the existence of a nonzero metric perturbation is not an invariant indication of the existence of a gravitational field. A better argument can be made if we note that the Riemann tensor has the first-order form

$$R^{\alpha}{}_{\eta\beta\lambda} = \frac{\epsilon}{2}(\gamma_{\alpha\lambda|\eta|\beta} + \gamma_{\beta\eta|\alpha|\lambda} - \gamma_{\alpha\beta|\eta|\lambda} - \gamma_{\eta\lambda|\alpha|\beta}) \tag{9.80}$$

as we found in Eq. (9.71). We know that the part of this tensor corresponding to the extraneous Weyl solution is identically zero, so by using Eq. (9.79a), we see that

$$\square^2 R^{\alpha}{}_{\eta\beta\lambda} = 0 \tag{9.81}$$

Thus the Riemann tensor, which gives an absolute criterion for the existence of a gravitational field, itself obeys the wave equation. It follows that, in the linearized theory, gravitational effects propagate with velocity c.

It should be noted carefully that the results of the above paragraph do not indicate whether or not gravitational radiation, which involves an energy transfer, exists. We may conclude only that the effects of gravity propagate at velocity c via the wave equation.

Let us now investigate the general properties of a plane-wave solution of Eqs. (9.79). It will be convenient, first, to study the general transformation properties of (9.79); specifically, we wish to obtain the conditions under which Eqs. (9.79) are invariant. For a general first-order coordinate transformation

$$\bar{x}^{\alpha} = x^{\alpha} - \epsilon\varphi_{\alpha}(x) \tag{9.82}$$

we found in the preceding section that, to first order in ϵ,

$$\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} = \delta^{\alpha}{}_{\mu} + \epsilon\varphi_{\alpha|\mu} \tag{9.83}$$

[Eq. (9.75)]. An easy calculation then gives

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - \epsilon(\varphi_{\mu|\nu} + \varphi_{\nu|\mu}) \tag{9.84}$$

or

$$\bar{\gamma}_{\mu\nu} = \gamma_{\mu\nu} - (\varphi_{\mu|\nu} + \varphi_{\nu|\mu}) \tag{9.85}$$

From (9.83) it is evident that $\partial/\partial\bar{x}^{\lambda} = \partial/\partial x^{\lambda} + O(\epsilon)$. Thus the D'Alembertian operator $\square^2$ is an invariant to zeroth order, and we can write (9.79a) in the barred system as

$$\bar{\square}^2\bar{\gamma}_{\mu\nu} = \square^2\gamma_{\mu\nu} - \square^2(\varphi_{\mu|\nu} + \varphi_{\nu|\mu}) \tag{9.86}$$

Similarly, (9.79b) becomes

$$\sum_{\beta=0}^{3}(\bar{\gamma}_{\eta\beta|\beta} - \tfrac{1}{2}\bar{\gamma}_{\beta\beta|\eta}) = \sum_{\beta=0}^{3}(\gamma_{\eta\beta|\beta} - \tfrac{1}{2}\gamma_{\beta\beta|\eta}) + \square^2\varphi_{\eta} \tag{9.87}$$

Thus it is clear that the linearized equations (9.79) will be invariant under the transformation (9.82) if and only if $\square^2\varphi_{\eta} = 0$. This fact is of interest by itself and will also be useful later in this section.

In the linearized theory, plane-wave solutions have the interesting property that the Riemann tensor is highly degenerate; by this we mean that not all the components of the metric tensor occur in $R^{\alpha}{}_{\eta\beta\lambda}$. We shall demonstrate this for a plane wave in the x direction. Such a wave is characterized by the fact that all variables $\gamma_{\alpha\beta}$ depend only on the coordinates $x^0 = ict$ and $x^1 = x$, so that the metric-tensor components have a vanishing y and z derivative. That is, $\gamma_{\mu\nu|2} = \gamma_{\mu\nu|3} = 0$. We use this condition and write down the 21 independent components of $R_{\alpha\eta\beta\lambda}$. Observe that, within our approximation, $R_{\alpha\eta\beta\lambda} = -R^{\alpha}{}_{\eta\beta\lambda}$. Since

$$R^{\alpha}{}_{\eta\beta\lambda} = \frac{\epsilon}{2}(\gamma_{\alpha\lambda|\eta|\beta} + \gamma_{\beta\eta|\alpha|\lambda} - \gamma_{\alpha\beta|\eta|\lambda} - \gamma_{\eta\lambda|\alpha|\beta}) \tag{9.88}$$

we obtain the following three groups of terms:

$$R^1{}_{223} = R^1{}_{323} = R^1{}_{023} = R^2{}_{323} = R^2{}_{320} = R^2{}_{330} = 0 \tag{9.89a}$$

$$\left\{\begin{array}{ll} R^1{}_{030} = \frac{\epsilon}{2}(\gamma_{30|1|0} - \gamma_{13|0|0}) & R^1{}_{020} = \frac{\epsilon}{2}(\gamma_{20|1|0} - \gamma_{12|0|0}) \\ R^1{}_{310} = \frac{\epsilon}{2}(\gamma_{31|1|0} - \gamma_{30|1|1}) & R^1{}_{210} = \frac{\epsilon}{2}(\gamma_{21|1|0} - \gamma_{20|1|1}) \\ R^1{}_{010} = \epsilon(\gamma_{10|0|1} - \frac{1}{2}\gamma_{11|0|0} - \frac{1}{2}\gamma_{00|1|1}) & \end{array}\right. \tag{9.89b}$$

$$\left\{\begin{array}{lll} R^2{}_{020} = -\frac{\epsilon}{2}\gamma_{22|0|0} & R^2{}_{030} = -\frac{\epsilon}{2}\gamma_{23|0|0} & R^3{}_{030} = -\frac{\epsilon}{2}\gamma_{33|0|0} \\ R^1{}_{220} = \frac{\epsilon}{2}\gamma_{22|0|1} & R^1{}_{230} = \frac{\epsilon}{2}\gamma_{23|0|1} & R^1{}_{320} = \frac{\epsilon}{2}\gamma_{32|0|1} \\ R^1{}_{330} = \frac{\epsilon}{2}\gamma_{33|0|1} & R^1{}_{212} = -\frac{\epsilon}{2}\gamma_{22|1|1} & R^1{}_{213} = -\frac{\epsilon}{2}\gamma_{23|1|1} \\ R^1{}_{313} = -\frac{\epsilon}{2}\gamma_{33|1|1} & & \end{array}\right. \tag{9.89c}$$

The values of all components $R^{\alpha}{}_{\eta\beta\lambda}$ can be obtained from these formulas by the above-mentioned approximate identity $R_{\alpha\eta\beta\lambda} = -R^{\alpha}{}_{\eta\beta\lambda}$ and the symmetries of the Riemann tensor. If we now impose the linearized field equations in the form $R_{\eta\lambda} = 0$, we obtain, for instance,

$$R_{13} = R^{\alpha}{}_{1\alpha 3} = R^{0}{}_{103} = 0 \tag{9.90}$$

A similar consideration of the remaining field equations yields the result that each of the components of the curvature tensor in group (9.89*b*) vanishes identically; thus only the components in group (9.89*c*) are nonzero. These components involve only γ_{22}, γ_{23}, γ_{32}, and γ_{33}; thus the curvature tensor is a function of only these elements of the metric tensor.

The formal result of the foregoing paragraph has a very interesting consequence. If we write the perturbation $\gamma_{\mu\nu}$ in two parts,

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}(1) + \gamma_{\mu\nu}(2)$$

$$\gamma_{\mu\nu}(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{22} & \gamma_{23} \\ 0 & 0 & \gamma_{32} & \gamma_{33} \end{pmatrix} \qquad \gamma_{\mu\nu}(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{03} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{20} & \gamma_{21} & 0 & 0 \\ \gamma_{30} & \gamma_{31} & 0 & 0 \end{pmatrix} \tag{9.91}$$

the curvature tensor of $\gamma_{\mu\nu}(2)$ will be identically zero. This indicates that there should exist a coordinate system where $\gamma_{\mu\nu}$ has only γ_{22}, γ_{23}, γ_{32}, and γ_{33} as nonzero components; that is, $\gamma_{\mu\nu}$ is a pure $\gamma_{\mu\nu}(1)$-type solution. Indeed such a coordinate system will be constructed in the following paragraphs. We shall refer to such a form as a canonical-wave solution.

To verify the above assertion we shall now explicitly solve (9.79) for a plane wave in the x^1 direction and put the result in canonical form by a coordinate transformation. Equation (9.79*a*) is automatically satisfied if we choose the plane-wave functional dependence

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}(x^1 - ct) = \gamma_{\mu\nu}(x^1 + ix^0)$$

Now denote $\sum_{\beta=0}^{3} \gamma_{\beta\beta} = \operatorname{Tr}\gamma$ by Γ, so the four components of (9.79*b*) may be written

$$\begin{aligned} \gamma_{00|0} + \gamma_{01|1} - \tfrac{1}{2}\Gamma_{|0} &= 0 \\ \gamma_{10|0} + \gamma_{11|1} - \tfrac{1}{2}\Gamma_{|1} &= 0 \\ \gamma_{20|0} + \gamma_{21|1} &= 0 \\ \gamma_{30|0} + \gamma_{31|1} &= 0 \end{aligned} \tag{9.92}$$

Because of the functional form of $\gamma_{\mu\nu}$, we can simplify these equations by noting that

$$\begin{aligned} \gamma_{\mu\nu|1} &= \gamma'_{\mu\nu} \\ \gamma_{\mu\nu|0} &= i\gamma'_{\mu\nu} \end{aligned} \tag{9.93}$$

where the prime denotes differentiation with respect to the argument $x^1 + ix^0$. Then we obtain for (9.92) the form

$$\begin{aligned} i\gamma'_{00} + \gamma'_{01} - \frac{i}{2}\Gamma' &= 0 \\ i\gamma'_{10} + \gamma'_{11} - \tfrac{1}{2}\Gamma' &= 0 \\ i\gamma'_{20} + \gamma'_{21} &= 0 \\ i\gamma'_{30} + \gamma'_{31} &= 0 \end{aligned} \tag{9.94}$$

Furthermore, since the $\gamma_{\mu\nu}$ all vanish at infinity for Euclidean boundary conditions, the entire set can be integrated at once merely by dropping the primes. Then multiplying the second equation by i and adding the first, we obtain

$$\gamma_{22} = -\gamma_{33} \qquad \Gamma = \operatorname{Tr}\gamma = \gamma_{00} + \gamma_{11} \tag{9.95}$$

Either the first or second equation in (9.94) may then be solved for γ_{01}:

$$\gamma_{01} = \frac{i}{2}(\gamma_{11} - \gamma_{00}) \tag{9.96}$$

Finally, the last two equations give

$$\gamma_{20} = i\gamma_{21} \qquad \gamma_{30} = i\gamma_{31} \tag{9.97}$$

Collecting the results (9.95) to (9.97), we write $\gamma_{\mu\nu}$ as

$$\gamma_{\mu\nu} = \left(\begin{array}{cc|cc} \gamma_{00} & \frac{i}{2}(\gamma_{11} - \gamma_{00}) & i\gamma_{12} & i\gamma_{13} \\ \frac{i}{2}(\gamma_{11} - \gamma_{00}) & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \hline i\gamma_{12} & \gamma_{12} & \gamma_{22} & \gamma_{23} \\ i\gamma_{13} & \gamma_{13} & \gamma_{23} & -\gamma_{22} \end{array}\right) \tag{9.98}$$

This is the most general solution in an arbitrary nearly Lorentzian coordinate system.

We wish now to put (9.98) in canonical form by going to a coordinate system where the extraneous components of (9.98) vanish, i.e., where $\gamma_{\mu\nu}$ has only a (2,3) subblock as in (9.91). In addition, we shall restrict ourselves to transformations of the form (9.82), with $\Box^2\varphi_\alpha = 0$, so that the linearized equations (9.79) retain their form in the barred system. If we demand that $\bar{\gamma}_{00} = \bar{\gamma}_{11} = \bar{\gamma}_{12} = \bar{\gamma}_{13} = 0$, then the transformation equation (9.85) implies that the φ_μ must obey the first-order equations

(9.99a) $$\varphi_{0|0} = \tfrac{1}{2}\gamma_{00}$$

(9.99b) $$\varphi_{1|1} = \tfrac{1}{2}\gamma_{11}$$

(9.99c) $$\varphi_{1|2} + \varphi_{2|1} = \gamma_{12}$$

(9.99d) $$\varphi_{1|3} + \varphi_{3|1} = \gamma_{13}$$

If we now choose the φ_μ to have the functional form $\varphi_\mu(x^1 + ix^0)$, the subsidiary condition $\Box^2\varphi_\mu = 0$ is clearly satisfied. In order to satisfy (9.99a), we need merely choose a function $F(Z)$ such that $F'(Z) = \gamma_{00}(Z)$; then $\varphi_0(x^1 + ix^0) = (-i/2)F(x^1 + ix^0)$ will satisfy (9.99a). Indeed, if we choose functions $F(Z)$, $G(Z)$, $H(Z)$, and $K(Z)$ such that

(9.100) $$F'(Z) = \gamma_{00}(Z) \qquad G'(Z) = \gamma_{11}(Z)$$ $$H'(Z) = \gamma_{12}(Z) \qquad K'(Z) = \gamma_{13}(Z)$$

then it is evident that Eqs. (9.99) and the subsidiary condition are satisfied by the following functions:

(9.101) $$\varphi_0 = \frac{-i}{2}F(x^1 + ix^0) \qquad \varphi_1 = \tfrac{1}{2}G(x^1 + ix^0)$$ $$\varphi_2 = H(x^1 + ix^0) \qquad \varphi_3 = K(x^1 + ix^0)$$

Note, furthermore, that the choice (9.101) leaves the (2,3) subblock unchanged because of the functional form of the φ_μ. This completes the demonstration that an allowed coordinate transformation always exists which will put $\gamma_{\mu\nu}$ into the canonical form

(9.102) $$\gamma_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_{22} & \gamma_{23} \\ 0 & 0 & \gamma_{23} & -\gamma_{22} \end{pmatrix}$$

To close this section we shall briefly study the motion of a test particle which moves in the metric field (9.102) of the plane wave. It describes

the geodesic motion

(9.103) $$\ddot{x}^\mu + \begin{Bmatrix} \mu \\ \alpha\ \beta \end{Bmatrix} \dot{x}^\alpha \dot{x}^\beta = 0$$

Because of the particularly simple structure of the metric (9.102), we can easily obtain the Christoffel symbols. We find that, to order ϵ,

(9.104) $$\begin{Bmatrix} 0 \\ \alpha\ \beta \end{Bmatrix} = -\frac{\epsilon}{2}(\gamma_{0\alpha|\beta} + \gamma_{0\beta|\alpha} - \gamma_{\alpha\beta|0}) = \frac{\epsilon}{2}\gamma_{\alpha\beta|0}$$

since all $\gamma_{0\alpha}$ are zero. Similarly,

(9.105) $$\begin{Bmatrix} 1 \\ \alpha\ \beta \end{Bmatrix} = \frac{\epsilon}{2}\gamma_{\alpha\beta|1}$$

On the other hand, we know that $\gamma_{\alpha\beta}$ is a function of $x^1 + ix^0$, so that $\gamma_{\alpha\beta|0} = i\gamma_{\alpha\beta|1}$, from which we obtain

(9.106) $$i\begin{Bmatrix} 1 \\ \alpha\ \beta \end{Bmatrix} = \begin{Bmatrix} 0 \\ \alpha\ \beta \end{Bmatrix}$$

From (9.103) and (9.106) it immediately follows that

(9.107) $$i\ddot{x}^0 + \ddot{x}^1 = 0$$

which has a first integral which will be useful:

(9.108) $$i\dot{x}^0 + \dot{x}^1 = A$$

Let us next display the equation of motion (9.103) for $\mu = 1$:

(9.109) $$\ddot{x}^1 + \frac{\epsilon}{2}[\gamma'_{22}((\dot{x}^2)^2 - (\dot{x}^3)^2) + 2\gamma'_{23}\dot{x}^2\dot{x}^3] = 0$$

We also desire the equations of motion for $\mu = 2$ and 3. Since

(9.110) $$\begin{Bmatrix} k \\ \alpha\ \beta \end{Bmatrix} = -\frac{\epsilon}{2}(\gamma_{\alpha k|\beta} + \gamma_{\beta k|\alpha}) \qquad k = 2, 3$$

we obtain

(9.111) $$\ddot{x}^k - \epsilon\gamma_{\alpha k|\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \qquad k = 2, 3$$

in which α runs only over 2 and 3, and β runs only over 0 and 1. Since $\gamma_{\alpha k|0} = i\gamma_{\alpha k|1} = i\gamma'_{\alpha k}$, we find

$$\ddot{x}^k = \epsilon\gamma'_{\alpha k}(i\dot{x}^0 + \dot{x}^1)\dot{x}^\alpha = \epsilon(i\dot{x}^0 + \dot{x}^1)(\gamma'_{2k}\dot{x}^2 + \gamma'_{3k}\dot{x}^3) \tag{9.112}$$

By virtue of (9.108), we therefore find

$$\ddot{x}^2 = \epsilon A(\gamma'_{22}\dot{x}^2 + \gamma'_{23}\dot{x}^3) \tag{9.113a}$$

$$\ddot{x}^3 = \epsilon A(\gamma'_{23}\dot{x}^2 + \gamma'_{33}\dot{x}^3) = \epsilon A(\gamma'_{23}\dot{x}^2 - \gamma'_{22}\dot{x}^3) \tag{9.113b}$$

The equations of motion (9.108), (9.109), and (9.113) allow us to analyze the nature of gravitational waves in a very enlightening way. We shall assume, as always, that the gravitational fields we deal with are weak and furthermore that the velocities of all particles are small compared to c. Then we have approximately

$$\dot{x}^0 = ic\frac{dt}{ds} \cong i \qquad \dot{x}^i \cong \frac{v^i}{c} \tag{9.114}$$

Equation (9.108) then reads

$$-1 + \frac{v^1}{c} \cong -A \tag{9.115}$$

so that v^1 must remain a constant. This is consistent with (9.109), which becomes

$$\ddot{x}^1 = -\frac{\epsilon}{2}\left[\gamma'_{22}\left(\left(\frac{v^2}{c}\right)^2 - \left(\frac{v^3}{c}\right)^2\right) + 2\gamma'_{23}\frac{v^2v^3}{c^2}\right] = O\left(\frac{\epsilon v^2}{c^2}\right) \tag{9.116}$$

so that $\ddot{x}^1$ is indeed quite small. Equations (9.113) become

$$\begin{aligned} \ddot{x}^2 &= \epsilon A(\gamma'_{22}\dot{x}^2 + \gamma'_{23}\dot{x}^3) = O\left(\frac{\epsilon v}{c}\right) \\ \ddot{x}^3 &= \epsilon A(\gamma'_{23}\dot{x}^2 + \gamma'_{33}\dot{x}^3) = O\left(\frac{\epsilon v}{c}\right) \end{aligned} \tag{9.117}$$

Thus to lowest order in the velocity there is no acceleration of the particle at all!

This conclusion does not, however, mean that particles are not *moved* by the gravitational wave, only that their *coordinates* do not change. In fact there is a definite physical displacement of two particles relative to

each other as a gravitational wave passes. To see this recall that the physical distance between two bodies whose coordinate separation is dx^i along the i direction is given by $\sqrt{g_{ii}}\,dx^i$, as we discussed in Sec. 4.2. Consider first the case of a wave for which $\gamma_{23} = 0$; we have a line element given by

$$ds^2 = c^2\,dt^2 - dx^2 - (1 - \epsilon\gamma_{22})\,dy^2 - (1 + \epsilon\gamma_{22})\,dz^2 \tag{9.118}$$

Thus the physical separations between test particles at rest in the coordinate system will change as the wave passes. For example, when $\gamma_{22} > 0$, the y separation decreases and the z separation increases, as shown in Fig. 9.1 for a group of four free test particles.

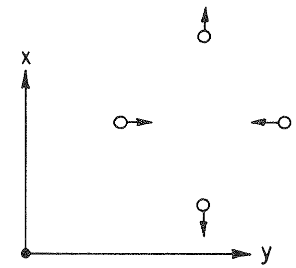

Fig. 9.1
A gravitational wave with $\gamma_{22} \neq 0$ and $\gamma_{23} = 0$ produces relative physical displacements in a group of particles as shown for the case of four test particles. A wave with $\gamma_{23} \neq 0$ and $\gamma_{22} = 0$ produces the same effect but with the axes rotated 45°.

A wave for which $\gamma_{22} = 0$ has a very beautiful relation to the above case for which $\gamma_{23} = 0$. The line element is

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 - 2\epsilon\gamma_{23}\,dy\,dz \tag{9.119}$$

We now perform a local coordinate rotation through an angle $\pi/4$ in the yz plane to get new coordinate intervals

$$d\bar{y} = \frac{1}{\sqrt{2}}(dy - dz) \qquad d\bar{z} = \frac{1}{\sqrt{2}}(dy + dz) \tag{9.120}$$

and a new line element

$$ds^2 = c^2\,dt^2 - dx^2 - (1 - \epsilon\gamma_{23})\,d\bar{y}^2 - (1 + \epsilon\gamma_{23})\,d\bar{z}^2 \tag{9.121}$$

Since this is precisely the same form as (9.118), we see that this second type of wave produces the same kind of effect as the first but with the axes rotated by 45°. Clearly any wave field is a superposition of these two types of waves.

The above situation is reminiscent of a similar situation in electrodynamics. A plane electromagnetic wave has two possible polarization directions, corresponding to two independent axes perpendicular to each other and to the direction of propagation of the wave. A charged particle at rest is accelerated by such a wave predominantly in the direction of its polarization. Thus the two polarization states of the electromagnetic wave produce accelerations that have the same magnitude but lie at right angles to each other. In the gravitational case we have a very similar situation, except that we must look not at the acceleration of a single particle but at the separation of two particles and the independent polarization states lie at 45° to each other.

It is important to note the fundamentally different nature of the motion of test bodies in the time-independent approximate metric of Sec. 4.3, which has nonzero γ_{00}, and the wave-type metric we have just studied, which has $\gamma_{00} = 0$. In the former case, motion resembles classical motion in a Newtonian field, but in classical gravitational theory there is no analogue of the field and the motion we have just studied: gravitational waves represent a qualitative difference between classical and relativistic gravitational theory. The difference between the metric fields is analogous to the difference between Coulomb and radiation fields in electromagnetic theory.

The detection of gravitational-wave pulses has been reported by Weber (1970a, 1970b). His detection apparatus consists of several rigid aluminum bars, each of which undergoes stresses and strains as a wave pulse passes over it. These bars are located many miles apart, and the signals are put in coincidence to reduce the contribution of noise. Although the displacements involved are very small, of order 10^{-14} cm, they are indirectly detectable through piezoelectric devices placed around the bar. A bar detector responds differently to waves coming from different directions and thus acts as a directional antenna. Early data of Weber suggested that the source of the pulses lay in the direction of the galactic center. Other workers have obtained negative results, detecting no pulses, apparently contradicting Weber's results (Levine and Garwin, 1973; Tyson, 1973). If future experiments were to confirm Weber's results, a difficult theoretical problem would arise in accounting for the origin of such a large number of intense pulses (Field et al., 1969). Some interesting sources of gravitational waves which have been studied theoretically are bodies in high-velocity orbits around black holes, rotating stars undergoing gravitational collapse, and supernova explosions, none of which appear to be capable of producing enough energy to be detected by any of the presently operating antennas. However, with a sufficient

increase in the sensitivity of gravitational-wave antennas we may expect the opening of a new field, gravitational astronomy.

Exercises

9.1 In the linearized theory solutions of the field equations may be superposed. That is, a linear combination of solutions is itself a solution. Superpose solutions of the form obtained in Sec. 9.2 to obtain new solutions for the following cases: (*a*) two point particles lying on the x axis, one at $x = d$ and the other at $x = -d$; (*b*) a continuous line of mass density $\rho(x)$ g/cm, extending from $x = -d$ to $x = d$.

9.2 Show that in the linearized theory any localized distribution of matter, e.g., contained within a sphere of radius d, has a metric which tends to (9.44) for $r \gg d$.

9.3 Obtain explicit solutions for the gravitational wave equations (9.79*a*) in which the metric components are of the form $\cos(\omega t - \mathbf{k} \cdot \mathbf{x})$ or $\sin(\omega t - \mathbf{k} \cdot \mathbf{x})$. Show that ω may be interpreted as the radian frequency of the waves and $\mathbf{k}$ as the propagation direction. What is the magnitude of $\mathbf{k}$? The subsidiary condition (9.79*b*) is guaranteed by the canonical form (9.102). Show that the general solution of the system (9.79) may be written as a superposition of such plane waves, integrated over frequencies, and summed over the two polarization states.

9.4 In Chap. 5 we related the components $R^i{}_{0j0}$ of the Riemann tensor to classical tidal forces, or derivatives of the Newtonian force. Use this correspondence to determine the force exerted on a rigid body by a gravitational wave.

9.5 Discuss how a simple system of masses and springs, e.g., two masses connected by a spring, would respond to a gravitational wave. Study the motion for a single sinusoidal wave as discussed in Exercise 9.3, in particular the response as a function of the wave frequency.

9.6 Apply the equation of geodesic deviation to the plane-wave metric to obtain the quadrupole deformations produced on a system of test particles by the two polarization states.

Problems

9.1 We have studied gravitational waves in vacuum without regard to their sources. This problem is discussed by Weber (1961) in detail (see also Sec. 10.5). Show that a system of moving masses will, in general, radiate gravitational waves, with a source strength proportional to the second time derivative of the quadrupole moment.

9.2 For a system with a sinusoidally varying quadrupole moment show that the power radiated is of order $P \sim \kappa^2 I^2 \omega^6/c^5$, where I is the moment of inertia and ω is the radian frequency. (An expression for the energy density of the gravitational field may be obtained from Sec. 10.2 or from Weber's book.)

9.3 Discuss the gravitational radiation produced when a rotating star of roughly solar mass collapses to a black hole (see Chap. 14). Use order-of-magnitude estimates.

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CHAPTER

10

The Gravitational Field Equations for Nonempty Space

In Chap. 5 we presented the gravitational field equations for free space,

$$R_{\gamma\eta} = 0 \tag{10.1}$$

which were first proposed by Einstein. Then, in Chap. 9, we found that these equations reduce in the limit of weak fields to Laplace's equation for the classical gravitational potential φ,

$$\sum_{i=1}^{3} \varphi_{|i|i} = 0 \tag{10.2}$$

and may therefore be considered to be a generalization of the classical theory for free space. In this chapter we shall make a similar generalization of the classical gravitational equation for nonempty space, Poisson's equation:

$$\sum_{i=1}^{3} \varphi_{|i|i} = 4\pi\rho\kappa \qquad \kappa = 6.67 \times 10^{-8} \text{ dyne-cm}^2/\text{g}^2 \tag{10.3}$$

The scalar ρ denotes the density of matter in space, and κ is the gravitational constant. Observe that a distribution of matter with density $\rho(x)$ in Euclidean space has the gravitational potential

$$\varphi(x) = -\kappa \int \frac{\rho(x')\, d^3x'}{|x - x'|} \tag{10.4}$$

and that (10.3) is a consequence of the identity

$$\nabla^2 \int \frac{\rho(x')}{|x - x'|}\, d^3x' = -4\pi\rho(x) \tag{10.5}$$

We shall find relativistic field equations for space containing matter (or energy) which have a tensor form analogous to (10.1) and reduce to (10.1) in the case of empty space. Using the relativistic field equations, we shall then investigate the classical limit for weak fields and show that (10.3) is a first approximation.

10.1 The Energy-Momentum Tensor

The classical equation (10.3) relates the behavior of the potential function φ to the density of *matter* in space ρ. In relativity theory, however, we cannot simply speak of the density of matter in space; we must also include the energy density, since, as Einstein has shown, matter and energy are indistinguishable with regard to their inertial properties: That is, $E = mc^2$. In dealing with nonempty space we shall therefore lump together the matter, radiant energy, elastic energy, etc., and speak of the *energy content* of space. This term, however, does *not* include gravitational energy.

We shall express the influence of matter and field energy in the form of a tensor $T^{\mu\nu}$ which is called the energy-momentum tensor. In this book we shall give $T^{\mu\nu}$ the dimensions of a pure mass density, grams per cubic centimeter. One could equally well give it the dimensions of energy density merely by multiplying the numerical values by c^2.

Following the development of the free-space field equations in Chap. 5 we shall seek a second-rank tensor equation of the general form

$$\begin{pmatrix}\text{Tensor representing}\\ \text{geometry of space}\end{pmatrix} = \begin{pmatrix}\text{tensor representing}\\ \text{energy content of space}\end{pmatrix} \tag{10.6}$$

These field equations must satisfy two limit requirements: they must be equivalent to *Poisson's equation* (10.3) in the *limit of weak fields* and must reduce to *Einstein's free-space field equations* (10.1) when the *energy density in space is zero*. We shall begin the development by considering in this section a class of tensors which describe the energy content of space and are suitable for use in the right side of the symbolic field equation (10.6). Since the free-space field equations (10.1) involve tensors of second rank, it is natural to limit our discussion to symmetric second-rank tensors.

Furthermore, we shall begin our work in a *flat* Riemann space and use the ideas of special relativity in order to retain simplicity until the end of this section. In order to develop machinery to describe matter in tensor form, we begin by recasting well-known equations of classical physics in tensor form and then look for common and characteristic features in the equations.

Let us first consider the simplest kind of energy field: a field of noninteracting incoherent matter. Such a field may be characterized by a scalar proper-density field $\rho_0(x)$ and a four-vector field of flow $u^\mu(x)$. Recall that the proper density is the density which would be measured by an observer moving *with* the flow. The four-velocity flow $u^\mu(x)$ is to be interpreted as follows: The element of matter which occupies the point x^μ of space-time has a motion $x^\mu(s)$ such that $dx^\mu/ds = u^\mu(x)$. Using these two characteristics of the matter field, the simplest second-rank tensor field we can construct is

$$T^{\mu\nu} = \rho_0(x)u^\mu(x)u^\nu(x) \tag{10.7}$$

We shall call this the energy-momentum tensor of the matter field or, more concisely, the matter tensor.

In order to obtain a physical interpretation of this tensor field, let us use the familiar coordinates of special relativity (ct,x,y,z) and the usual Lorentz metric tensor. The component T^{00} of the matter tensor can be written in terms of the motion of the particle of matter occupying the point x^μ:

$$T^{00} = \rho_0 \frac{dx^0}{ds}\frac{dx^0}{ds} = c^2\rho_0\left(\frac{dt}{ds}\right)^2 \tag{10.8}$$

Consider now a *particle* of matter in the field which is moving with some three-dimensional velocity $\mathbf{v}$. Since we may use its time coordinate as a trajectory parameter instead of s, and since our metric at present is

$$ds^2 = c^2\,dt^2 - (dx^2 + dy^2 + dz^2) = c^2\,dt^2\left(1 - \frac{v^2}{c^2}\right) \tag{10.9}$$

we have

$$\frac{ds}{dt} = c\left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{10.10}$$

If we call $(1 - v^2/c^2)^{-1/2} = \gamma$ (which is always ≥ 1), we can write dt/ds as γ/c; the component T^{00} can then be expressed concisely as

$$T^{00} = \gamma^2\rho_0 \tag{10.11}$$

This has a simple physical interpretation; in special relativity the mass of a volume of moving material increases by a factor γ over its rest mass, while a moving three-dimensional volume element appears to have decreased in volume by the same factor. Thus, from the point of view of a fixed observer, the *density increases* by a factor γ^2. Hence, if a field of material of proper density ρ_0 flows past a fixed observer at a velocity $\mathbf{v}$, the observer will measure a density $\rho = \gamma^2\rho_0$. The component c^2T^{00} may therefore be interpreted as the *relativistic energy density* of the matter field since the only contribution to the energy of the field is due to motion of the matter.

The other components of $T^{\mu\nu}$ are also easily interpreted. Let us consider the space coordinates (x,y,z) of a particle in the field to be functions of the zeroth coordinate, time. Then the mixed space-time components are

$$T^{0i} = \rho_0 \frac{dx^0}{ds}\frac{dx^i}{ds} = \rho_0 c \left(\frac{dt}{ds}\right)\left(\frac{dx^i}{dt}\right)\left(\frac{dt}{ds}\right) \tag{10.12}$$

If we denote the three-dimensional velocity dx^i/dt by v^i, this gives

$$T^{0i} = \rho_0\gamma^2 \frac{v^i}{c} = \rho\frac{v^i}{c} \tag{10.13}$$

Similarly, the space-space components of $T^{\mu\nu}$ are given by

$$T^{ij} = \rho_0 \frac{dx^i}{ds}\frac{dx^j}{ds} = \rho_0\gamma^2\frac{v^iv^j}{c^2} = \rho\frac{v^iv^j}{c^2} \tag{10.14}$$

so the entire matter tensor can be displayed as

$$T^{\mu\nu} = \rho\begin{pmatrix} 1 & v_x/c & v_y/c & v_z/c \\ v_x/c & v_x^2/c^2 & v_xv_y/c^2 & v_xv_z/c^2 \\ v_y/c & v_yv_x/c^2 & v_y^2/c^2 & v_yv_z/c^2 \\ v_z/c & v_zv_x/c^2 & v_zv_y/c^2 & v_z^2/c^2 \end{pmatrix} \tag{10.15}$$

Let us note at this point that, just as the metric tensor corresponds to the classical gravitational *potential* according to the approximate relation $g_{00} \cong 1 + 2\varphi/c^2$, so the energy-momentum tensor corresponds to the *density* of energy in space according to the approximate relation $T^{00} \cong \rho_0$. In both cases the validity of the approximation depends upon weak fields and low velocities, $|\mathbf{v}| \ll c$.

The matter tensor can be used to write the special relativistic equations

of force-free motion for a matter field in a very elegant way. To demonstrate this let us compute the zeroth component of the divergence of $T^{\mu\nu}$:

$$T^{0\nu}{}_{|\nu} = \frac{1}{c}\frac{\partial\rho}{\partial t} + \frac{1}{c}\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] \tag{10.16}$$

In three-dimensional vector notation this is

$$T^{0\nu}{}_{|\nu} = \frac{1}{c}\left[\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{v}\right] \tag{10.17}$$

The right side of this equation is familiar from the continuity equation of classical hydrodynamics:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{v} = 0 \tag{10.18}$$

This well-known kinematic relation expresses quite generally the conservation of a quantity of material with density ρ moving with a velocity field $\mathbf{v}$. Here it expresses the conservation of matter in the sense of special relativity, which is the same as the conservation of energy. It follows that the *conservation of energy* in a free-flowing matter field is expressible as

$$T^{0\nu}{}_{|\nu} = 0 \tag{10.19}$$

The next term of the divergence of $T^{\mu\nu}$ is

$$\begin{aligned} T^{1\nu}{}_{|\nu} &= \frac{1}{c^2}\left[\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x^2)}{\partial x} + \frac{\partial(\rho v_xv_y)}{\partial y} + \frac{\partial(\rho v_xv_z)}{\partial z}\right] \\ &= \frac{\rho}{c^2}\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] \\ &\quad + \frac{v_x}{c^2}\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] \end{aligned} \tag{10.20}$$

The second term of this expression is zero by the conservation-of-energy equation (10.18); the remainder can then be written as

$$T^{1\nu}{}_{|\nu} = \frac{\rho}{c^2}\left(\frac{\partial v_x}{\partial t} + \mathbf{v}\cdot\nabla v_x\right) \tag{10.21}$$

Similarly, the remaining terms of the divergence may be included in

$$T^{i\nu}{}_{|\nu} = \frac{\rho}{c^2}\left[\frac{\partial v^i}{\partial t} + \mathbf{v}\cdot\nabla v^i\right] \tag{10.22}$$

The right side of this expression is also familiar from hydrodynamics; the force-free motion of a field of material can be described by setting the Eulerian derivative or flow derivative equal to zero. The Eulerian derivative of any quantity $Q(x)$ is the change of Q as it would appear to an observer following a streaming particle; that is,

$$\frac{DQ}{Dt} \equiv \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x^i} v^i = \frac{\partial Q}{\partial t} + \mathbf{v} \cdot \nabla Q \tag{10.23}$$

For the present case we therefore have, by setting the Eulerian derivative of the matter flow equal to zero,

$$\frac{Dv^i}{Dt} = \frac{\partial v^i}{\partial t} + \mathbf{v} \cdot \nabla v^i = 0 \tag{10.24}$$

It then follows that the equations of force-free hydrodynamic flow of the matter field can be written as

$$T^{i\nu}{}_{|\nu} = 0 \tag{10.25}$$

The Euler equations (10.24) are derived from and are equivalent to the principle of conservation of momentum. Thus, from (10.19) and (10.25), we see that demanding that the energy-momentum tensor have zero divergence is equivalent to demanding conservation of energy and conservation of momentum in the matter field.

We have described the motion of incoherent matter by the tensor law contained in (10.25) and (10.19). The mathematical advantage of this formulation comes from the fact that we can now translate this law into any coordinate system. It must take the form

$$T^{\mu\nu}{}_{\|\nu} = 0 \tag{10.26}$$

on purely formal grounds of covariance. Clearly, this law has been established only in the case that the matter moves in a Lorentzian space, i.e., a flat Riemann space. Here it expresses the conservation of energy and momentum of the matter field during the flow. We shall discuss the case of a general (nonflat) Riemann space after we have considered a few more examples of energy-momentum tensors in flat space.

10.2 Inclusion of Forces in $T^{\mu\nu}$

In the above paragraphs we have considered an incoherent matter field on which *no forces* act and whose particles do not interact. We shall now show what sort of modifications are to be made when an internal force such as pressure is present. Specifically, we shall consider a perfect fluid which is by definition characterized by a proper-density field $\rho_0(x)$, a four-vector velocity field of flow $u^\mu(x)$, and a scalar pressure field $p(x)$. We shall show that, by adding an appropriate term to the material energy-momentum tensor which we now denote as $M^{\mu\nu}$,

$$M^{\mu\nu} = \rho_0 u^\mu(x) u^\nu(x) \tag{10.27}$$

the effect of the internal force can be included in the framework of the preceding paragraphs; i.e., setting the divergence of an appropriate complete energy-momentum tensor equal to zero will give the correct equations of motion and express the conservation of energy.

We wish at first to work in the classical limit with low fluid velocities and low pressure, so we shall neglect terms of order v^2/c^2 and $p(v/c)$. Furthermore, we shall assume that the pressure is sufficiently small so that the elastic energy density of the fluid can be neglected in comparison with the energy due to material density. With these assumptions we can write the conservation of energy completely in terms of the proper matter density ρ_0:

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) = 0 \tag{10.28}$$

As is well known from fluid dynamics, the equations of motion [Eqs. (10.24)] now contain a volume force equal to the negative of the pressure gradient $\partial p/\partial x^i$:

$$\rho_0 \frac{Dv^i}{Dt} = \rho_0 \left(\frac{\partial v^i}{\partial t} + \mathbf{v} \cdot \nabla v^i \right) = f^i = -\frac{\partial p}{\partial x^i} \tag{10.29}$$

The meaning of this equation is evident. The volume element is accelerated by the pressure-force density $-\partial p/\partial x^i$, and an observer moving with the fluid experiences the acceleration Dv^i/Dt. Thus (10.29) is merely Newton's second law of dynamics. In the case of exterior volume forces, the term f^i would naturally have to be modified. Clearly, the equation is not yet relativistically invariant. It will be the task of the tensor formulation to adjust the approximate classical equations to the demands of relativistic covariance.

If we denote the matter tensor $\rho_0 u^\mu u^\nu$ as displayed in (10.15) by $M^{\mu\nu}$ to avoid confusion with the complete energy-momentum tensor $T^{\mu\nu}$, we can express the conservation of energy by

$$M^{0\nu}{}_{|\nu} = \frac{1}{c} \left(\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) \right) = 0 \tag{10.30}$$

just as in the case of the incoherent matter field of the preceding example. The equations of motion (10.29), however, must now be expressed as

$$(10.31)\qquad M^{i\nu}{}_{|\nu} = \frac{\rho_0}{c^2}\frac{Dv^i}{Dt} = \frac{\rho_0}{c^2}\left(\frac{\partial v^i}{\partial t} + \mathbf{v}\cdot\nabla v^i\right) = -\frac{1}{c^2}\frac{\partial p}{\partial x^i} \neq 0$$

Thus $M^{\mu\nu}$ is not divergenceless because of the presence of the internal pressure force. To remedy this let us consider a 3×3 matrix S^{ij} with the property that

$$(10.32)\qquad S^{ij}{}_{|j} = \frac{1}{c^2}\frac{\partial p}{\partial x^i}$$

Such a matrix is known as a three-dimensional stress tensor; its divergence represents a force. A stress tensor satisfying (10.32) is easily seen to be

$$(10.33)\qquad S^{ij} = \frac{p}{c^2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Using this pressure stress tensor, we may write the equations of motion (10.31) as

$$(10.34)\qquad M^{i\nu}{}_{|\nu} + S^{ij}{}_{|j} = 0$$

Indeed, if we extend S^{ij} into a 4×4 matrix

$$(10.35)\qquad S^{\mu\nu} = \frac{p}{c^2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

we may combine the conservation-of-energy equation (10.30) and the equations of motion (10.31) into a single matrix equation

$$(10.36)\qquad (M^{\mu\nu} + S^{\mu\nu})_{|\nu} = T^{\mu\nu}{}_{|\nu} = 0$$

where $T^{\mu\nu}$ is explicitly

$$(10.37)\qquad T^{\mu\nu} = \rho_0\begin{pmatrix}1 & v_x/c & v_y/c & v_z/c\\ v_x/c & 0 & 0 & 0\\ v_y/c & 0 & 0 & 0\\ v_z/c & 0 & 0 & 0\end{pmatrix} + \frac{p}{c^2}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

within our approximation, which represents the classical limit. Note, however, that $T^{\mu\nu}$ is clearly not a tensor since the relation between $\mathbf{v}$, p, and $T^{\mu\nu}$ is not covariant.

What we wish to do now is generalize (10.37) so that it becomes a true tensor. The material term $M^{\mu\nu}$ raises no problem since it is already expressible as a tensor as in (10.7), so we have only to find the proper form for $S^{\mu\nu}$. This term can be extended into an actual tensor by noting that there are only two second-rank symmetric tensor fields associated with the fluid, $g^{\mu\nu}$ and $u^\mu u^\nu$. Thus the extended $S^{\mu\nu}$ must be a linear combination of the form

$$(10.38)\qquad S^{\mu\nu} = \frac{p}{c^2}[\alpha u^\mu u^\nu + \beta g^{\mu\nu}]$$

which has to reduce to (10.37) for low velocities and pressure. It is easily seen that the neglect of v^2/c^2 and $p(v/c)$ terms in (10.38) leads to

$$(10.39)\qquad S^{\mu\nu} = \frac{p}{c^2}\left[\alpha\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + \beta\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\right]$$

so the choice $\alpha = 1$, $\beta = -1$ evidently completes the task of generalizing $S^{\mu\nu}$ to a true tensor; we are thus led to

$$(10.40)\qquad S^{\mu\nu} = \frac{p}{c^2}(u^\mu u^\nu - g^{\mu\nu})$$

and the complete energy-momentum tensor

$$(10.41)\qquad T^{\mu\nu} = \rho_0 u^\mu u^\nu + \frac{p}{c^2}(u^\mu u^\nu - g^{\mu\nu})$$

By correspondence with the classical matrix equation (10.36) we assume that this complete energy-momentum tensor has zero divergence in a flat Riemann space:

$$(10.42)\qquad T^{\mu\nu}{}_{||\nu} = 0$$

This elegant formula is the covariant formulation of the flow of a fluid under the effect of its own internal pressure force.

In order to understand better the significance of the terms in the complete tensor $T^{\mu\nu}$, let us integrate the equation $T^{0\nu}{}_{||\nu} = 0$ over a fixed part of three-space V with boundary Σ. We have, in the usual coordinates of special relativity for which covariant and ordinary differentiation are identical,

(10.43) $$\int_V T^{0\nu}{}_{|\nu}\, dV = \frac{1}{c}\frac{\partial}{\partial t}\int_V T^{00}\, dV + \int_\Sigma T^{0i} n_i\, d\sigma = 0$$

Here n_i is the exterior normal three-vector to the surface element $d\sigma$ of Σ. This relation is in the form of a conservation law. If we recall that, in the case of incoherent matter flow, the term $c^2M^{00} = c^2\rho$ represented the energy density and $cM^{0i} = \rho v^i$ represented the momentum density, it is natural to identify the corresponding terms of the $S^{\mu\nu}$ tensor as follows:

(10.44) $$c^2S^{00} = p(\gamma^2 - 1) = p\frac{v^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

is the pressure energy-density of the fluid flow, which is in general a small quantity. Similarly, cS^{0i} is the momentum density due to the fluid pressure. The total energy density is thus $c^2(M^{00} + S^{00}) = c^2T^{00}$, and the total momentum density is $c(M^{0i} + S^{0i}) = cT^{0i}$. Equation (10.43) expresses the fact that energy changes in V are caused by transport of momentum through the boundary Σ. It is convenient, therefore, to display the entire energy-momentum tensor as

(10.45) $$T^{\mu\nu} = M^{\mu\nu} + \begin{pmatrix} h & \mathbf{g} \\ \mathbf{g} & S \end{pmatrix}$$

where c^2h and $c\mathbf{g}$ represent the energy density and momentum densities associated with the pressure, and S is the three-dimensional stress tensor.

The value of the tensor law (10.42) is twofold. First, we have obtained the relativistic laws of fluid dynamics in a Lorentz covariant form which differs for high pressures and velocities from the classical noncovariant form. Second, we have a tensor law which automatically holds in all curvilinear coordinate systems. Such an elegant situation is highly desirable.

10.3 The Electromagnetic Field and $T^{\mu\nu}$

As another example involving internal forces, let us consider a flowing field of charged matter which is described by a proper density $\rho_0(x)$, a four-vector velocity u^μ, and a proper electric charge density $\sigma_0(x)$. We are still working in the flat Riemann space of special relativity, so by using the usual coordinates ct, x, y, z, we can write Maxwell's equations as

(10.46) $$F^{\mu\nu}{}_{|\nu} = s^\mu \qquad \{F_{\mu\nu|\lambda}\} = 0$$

The source vector s^μ is associated with the motion and charge of the matter field; it is in fact related to the charge density and four-velocity by

(10.47) $$s^\mu = \sigma_0 u^\mu$$

This relation is easily verified if we consider the space coordinates x^i of a particle of matter which occupies the point x^μ and moves with the flow. The x^i can be considered to be functions of the time coordinate ct, for then we may write $\sigma_0 u^\mu$ as

(10.48) $$\sigma_0\frac{dx^\mu}{ds} = \sigma_0 c\frac{dt}{ds}\left(1, \frac{1}{c}\frac{dx}{dt}, \frac{1}{c}\frac{dy}{dt}, \frac{1}{c}\frac{dz}{dt}\right) = \sigma_0 c\frac{dt}{ds}\left(1, \frac{\mathbf{v}}{c}\right)$$

Using the relation (10.10) between dt and ds for a moving particle of matter, we have

(10.49) $$\sigma_0\frac{dx^\mu}{ds} = \left(\gamma\sigma_0, \gamma\sigma_0\frac{\mathbf{v}}{c}\right)$$

It is well known from special relativity that, because of the shrinking of a moving volume element by a factor γ, the charge density σ measured by a stationary observer is increased by a factor γ; that is, $\sigma = \gamma\sigma_0$. Using this fact and noting that the convection current of the charged matter field is $\mathbf{j} = \sigma\mathbf{v}$, we can write (10.49) as

(10.50) $$\sigma_0\frac{dx^\mu}{ds} = \left(\sigma, \frac{\mathbf{j}}{c}\right)$$

which is by definition the source vector s^μ [Eq. (4.44)]. This verifies (10.47) and allows us to write the Maxwell equations (10.46) as

(10.51) $$F^{\mu\nu}{}_{|\nu} = \sigma_0 u^\mu \qquad \{F_{\mu\nu|\lambda}\} = 0$$

These equations completely determine the behavior of the electromagnetic field in terms of the behavior of the charged matter.

Let us next consider the converse problem, the behavior of the charged matter under the influence of the electromagnetic field. For convenience we shall begin working in the classical limit of small matter-velocity and neglect terms of order v^2/c^2; we shall also consider the charge density σ_0 to be sufficiently small so that we may neglect the density of electromagnetic energy. Under these assumptions the conservation-of-energy equation involves only the proper matter density ρ_0:

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot \rho_0 \mathbf{v} = 0 \tag{10.52}$$

This may also be expressed using the material tensor $M^{\mu\nu}$ as

$$M^{0\nu}{}_{|\nu} = 0 \tag{10.53}$$

precisely as in the case of the perfect-fluid example. The equations of motion of the charged material are

$$\rho_0 \frac{Dv^i}{Dt} = \rho_0 \left\{ \frac{\partial v^i}{\partial t} + \mathbf{v} \cdot \nabla v^i \right\} = f^i \tag{10.54}$$

where f^i is the Lorentz force:

$$f^i = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H} \right)^i \tag{10.55}$$

It can be easily verified that f^i may be expressed in terms of the tensor $F^{\mu\nu}$ as

$$f^i = -\sigma_0 F^{i\nu} u_\nu \tag{10.56}$$

where, because of the Lorentz metric, $u_0 = u^0$ and $u_i = -u^i$. To show this we write out the component $i = 1$; since $u^\nu = dx^\nu/ds$ and $u^i = u^0 v^i/c$, where v^i is of course the ordinary velocity, we have

$$-\sigma_0 F^{1\nu} u_\nu = -\sigma_0 u_0 \left(F^{10} - F^{12} \frac{v_y}{c} - F^{13} \frac{v_z}{c} \right) \tag{10.57}$$

Let us note that $u_0 = dx_0/ds = \gamma$ from (10.10), and the Lorentz volume-contraction factor is γ, so the relativistic charge density is $\sigma = \gamma\sigma_0$. Using this fact and the explicit form (4.48) for $F^{\mu\nu}$, we have

$$-\sigma_0 F^{1\nu} u_\nu = \sigma_0 \gamma \left(E_x + H_z \frac{v_y}{c} - H_y \frac{v_z}{c} \right) = \sigma \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H} \right)_x \tag{10.58}$$

Analogous formulas hold for the other components, which verifies (10.56). Thus we may write the equations of motion of the charged matter as

$$\rho_0 \frac{Dv^i}{Dt} = -\sigma_0 F^{i\nu} u_\nu \tag{10.59}$$

or by virtue of the identity (10.22),

$$M^{i\nu}{}_{|\nu} = -\frac{\sigma_0}{c^2} F^{i\nu} u_\nu \qquad M^{\mu\nu} = \rho_0 u^\mu u^\nu \tag{10.60}$$

By analogy with the analysis of the perfect fluid, we therefore wish to obtain a matrix S with the property

$$S^{i\nu}{}_{|\nu} = \frac{\sigma_0}{c^2} F^{i\nu} u_\nu \tag{10.61}$$

so that $M^{i\nu} + S^{i\nu}$ will be divergenceless by virtue of (10.60). In order to do this we replace $i = 1, 2, 3$ by $\mu = 0, 1, 2, 3$ and lower the index in (10.61). This gives

$$S_\mu{}^\nu{}_{|\nu} = \frac{\sigma_0}{c^2} F_\mu{}^\nu u_\nu \tag{10.62}$$

Allowing $\mu = 0$ in this, we should violate the conservation law (10.52), but for small velocities and fields, this is a negligible correction. For large fields, we expect an electromagnetic correction to the energy density.

By using (10.51) we can substitute $(1/\sigma_0) F_\nu{}^\lambda{}_{|\lambda}$ for u_ν to rewrite (10.62) in the form

$$c^2 S_\mu{}^\nu{}_{|\nu} = F_\mu{}^\nu F_\nu{}^\lambda{}_{|\lambda} \tag{10.63}$$

Evidently, then, $S_\mu{}^\nu$ must be bilinear in the tensor $F^{\mu\nu}$; the most general bilinear tensor form of this kind is

$$c^2 S_\mu{}^\nu = A F_{\mu\alpha} F^{\alpha\nu} + B g_\mu{}^\nu F^{\alpha\beta} F_{\alpha\beta} \tag{10.64}$$

where A and B are constants. To determine the constants A and B, we take the divergence of $S_\mu{}^\nu$ and obtain

$$c^2 S_\mu{}^\nu{}_{|\nu} = A F_{\mu\alpha} F^{\alpha\nu}{}_{|\nu} + A F_{\mu\alpha|\nu} F^{\alpha\nu} + 2B g_\mu{}^\nu F^{\alpha\beta} F_{\alpha\beta|\nu} \tag{10.65}$$

Relabeling dummy indices and rearranging terms, we have

$$c^2 S_\mu{}^\nu{}_{|\nu} = A F_\mu{}^\nu F_\nu{}^\lambda{}_{|\lambda} + F^{\alpha\beta}(A F_{\mu\alpha|\beta} + 2B F_{\alpha\beta|\mu}) \tag{10.66}$$

This will be the desired result (10.63) if we set $A = 1$ and choose B so that the second term is zero. This is easily done, for the second term